ON \( p \)-ABSOLUTELY SUMMING CONSTANTS OF BANACH SPACES*

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ABSTRACT

Given \( 1 \leq p < \infty \) and a real Banach space \( X \), we define the \( p \)-absolutely summing constant \( \mu_p(X) \) as

\[
\inf \{ \sup \{ \sum_{i=1}^{m} |x^*(x_i)|^p / \sum_{i=1}^{m} \|x_i\|^p \} \},
\]

where the supremum ranges over \( \{x^* \in X^*; \|x^*\| \leq 1 \} \) and the infimum is taken over all sets \( \{x_1, x_2, \ldots, x_m\} \subseteq X \) such that \( \sum_{i=1}^{m} \|x_i\| > 0 \). It follows immediately from [2] that \( \mu_p(X) > 0 \) if and only if \( X \) is finite dimensional. In this paper we find the exact values of \( \mu_p(X) \) for various spaces, and obtain some asymptotic estimates of \( \mu_p(X) \) for general finite dimensional Banach spaces.

1. Preliminaries and definitions. The results obtained here are in part related to those in [3]. We recall briefly some basic definitions: the projection constant of a Banach space \( X \) is defined as \( \lambda(X) = \inf \{ \lambda > 0; \text{from every Banach space } Y \supset X, \text{there is a projection onto } X \text{ with norm } \leq \lambda \} \). The Macphail constant is defined as \( \mu(X) = \inf \{ \sup_{J} \| \sum_{j \in J} x_j \| / \sum_{j \in J} \| x_j \| \}, \) where \( J \) ranges on the subsets of \( \{1, 2, \ldots, m\} \), and the infimum is taken over all finite sets \( \{x_1, x_2, \ldots, x_m\} \subseteq X \) such that \( \sum_{j=1}^{m} \|x_j\| > 0 \). The distance \( d(X, Y) \) between isomorphic Banach spaces \( X \) and \( Y \) is defined as \( \inf \{ \| T \| \| T^{-1} \| ; T \text{ is an isomorphism of } X \text{ onto } Y \} \).

\( l_n^p (1 \leq p < \infty) \) denotes the space of real \( n \)-tuples \( x = (x_1, x_2, \ldots, x_n) \) with the norm \( \| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} \); \( l_n^{\infty} \) denotes the same space with the norm \( \| x \|_{\infty} = \max |x_i| \). All the asymptotic values of \( \mu(l_n^p) \), \( \lambda(l_n^p) \), \( d(l_n^p, l_m^q) (1 \leq p, q \leq \infty) \) are now known (see [3] for a short summary).

2. Formulation of results. We state here the main results which are to be proved.

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THEOREM 1. Given a Banach space $X$, let $K^*$ be the $\omega^*$ closure of the set of all the extremal points of the unit ball of $X^*$. For any $1 \leq p < \infty$, there is a probability measure (i.e. a regular non-negative Borel measure with total mass 1) $\nu$ over $K^*$, such that

$\mu_p(X) = \inf_{\|x\| = 1} \left( \int_{K^*} |x^*(x)|^p d\nu(x^*) \right)^{1/p}$. \hfill (1)

Moreover, for every probability measure $\theta$ over $K^*$ and every $1 \leq p < \infty$, the following inequality holds

$\mu_p(X) \geq \inf_{\|x\| = 1} \left( \int_{K^*} |x^*(x)|^p d\theta(x^*) \right)^{1/p}$. \hfill (2)

THEOREM 2. If $1 \leq p < \infty$, then

$\mu_p(l_2^n) = n^{-1/p}$, \hfill (3)

$\mu_p(l_1^n) = \left[ \frac{\Gamma(n/2)\Gamma(p/2 + 1/2)}{\Gamma(1/2)\Gamma(n/2 + p/2)} \right]^{1/p}$, \hfill (4)

$\mu_p(l_\infty^n) = \left[ 2^{-n} n^{-p} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) |n - 2k|^p \right]^{1/p}$, \hfill (5)

$\mu_p(G_{2n}) = \left[ (2n)^{-1} \sum_{i=0}^{2n-1} \left| \cos(i\pi/n) \right|^p \right]^{1/p}$, \hfill (6)

where $G_{2n}$ is the space whose unit ball is the affine-regular $2n$-sided polygon in the Minkowsky plane.

THEOREM 3. Let $X$ be an $n$-dimensional real Banach space, then

$\mu_p(X) \leq n^{-1/4}$, if $1 \leq p \leq 2$, \hfill (7)

$\mu_p(X) \leq c_p n^{-1/(2+p)}$, if $2 \leq p < \infty$, \hfill (8)

$\mu_p(X) \mu_p(X^*) \leq c_p n^{-2/3}$, if $1 \leq p \leq 2$, \hfill (9)

$\mu_p(X) \mu_p(X^*) \leq c_p n^{-4/(4+p)}$, if $2 \leq p \leq 4$, \hfill (10)

$\mu_p(X) \mu_p(X^*) \leq c_p n^{-1/2}$, if $4 \leq p < \infty$, \hfill (11)

($c_p$ denotes a constant depending only on $p$).

THEOREM 4. Let $X$ be an $n$-dimensional real Banach space, and $K_G$ the universal Grothendieck constant ($\pi/2 \leq K_G \leq \sinh(\pi/2)$). Then