INDECOMPOSABLE CONVEX POLYTOPES

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ABSTRACT
Shephard has given a criterion for the indecomposability (in the sense of Minkowski addition) of a convex polytope, in terms of strong chains of indecomposable faces joining pairs of vertices. Here, this criterion is weakened, to one involving strongly connected sets of indecomposable faces meeting every facet.

1. A criterion for indecomposability

In [3], Shephard gave a criterion for the indecomposability of a convex polytope with respect to Minkowski addition. We recall that a convex polytope $P$ in euclidean space $\mathbb{E}^d$ is said to be indecomposable if, in any expression $P = Q + R = \{x + y \mid x \in Q, y \in R\}$, each summand $Q$ is homothetic to $P$ or a point, so that $Q = \lambda P + t$ for some $\lambda \geq 0$ and $t \in \mathbb{E}^d$ (and then $0 \leq \lambda \leq 1$ and $R = (1 - \lambda)P - t$). Throughout, we shall follow the terminology of [1].

Shephard's criterion can be phrased succinctly using the concept of a strong chain of faces of a polytope $P$, which is a sequence $F_0, F_1, \ldots, F_k$ of faces such that $\dim(F_{j-1} \cap F_j) \geq 1$ for $j = 1, \ldots, k$. Such a chain joins two vertices $u$ and $v$ of $P$ if (say) $u \in F_0$ and $v \in F_k$. Shephard's result is

**Theorem 1.** A convex polytope, any two of whose vertices can be joined by a strong chain of indecomposable faces, is itself indecomposable.

A family $\mathcal{F}$ of faces of a polytope $P$ is called strongly connected if for each $F, G \in \mathcal{F}$, there exists a strong chain $F = F_0, F_1, \ldots, F_k = G$ with each $F_j \in \mathcal{F}$. A subset $\mathcal{F}$ of faces touches a face $F$ of $P$ if $(\bigcup \mathcal{F}) \cap F \neq \emptyset$. Recalling that a facet of $P$ is a face $F$ of dimension $\dim F = \dim P - 1$, our new criterion is given by

**Theorem 2.** If a polytope has a strongly connected family of indecomposable faces which touches each of its facets, then it is itself indecomposable.

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In fact, Theorems 1 and 2 are not strictly comparable; however, in all applications of Theorem 1 which we are aware of, the strong chains fit together to form a strongly connected set.

2. Proof of the theorem

Without loss of generality we consider a $d$-polytope $P$ in $E^d$, which possesses a strongly connected family $\mathcal{F}$ of indecomposable faces which touches every facet. We can express $P$ in the form

$$P = \{ x \in E^d \mid \langle x, u_i \rangle \leq \eta_i \ (i = 1, \ldots, n) \},$$

where $u_1, \ldots, u_n$ are the outer normal vectors to the facets of $P$. Now, a summand $Q$ of $P$ is of the same form, with $\eta_i$ replaced by $\zeta_i$ (say) for $i = 1, \ldots, n$. In general, the $u_i$ are not all now facet normals of $Q$; in the present case, however, we must show that they are (unless $Q$ is a point), and, indeed, that $\zeta_i = \lambda \eta_i + \langle t, u_i \rangle \ (i = 1, \ldots, n)$ for some $\lambda \geq 0$ and $t \in E^d$, which corresponds to the relation $Q = \lambda P + t$. For further details, see [1], Chapter 14, or [2].

If $u$ is any non-zero vector, we write $P_u$ for the face of $P$ in direction $u$, that is, the intersection of $P$ with its support hyperplane with outer normal $u$. Then $P = Q + R$ implies $P_u = Q_u + R_u$.

Now consider any strong chain $F_i, F_i, \ldots, F_k$ of indecomposable faces of $P$, and let $G_i, G_i, \ldots, G_k$ be the corresponding chain of faces of its summand $Q$. Since each $F_i = P_u$ for some $u_i$, and since $F_i$ is indecomposable, it follows that $G_i = \lambda_i F_i + t_i$ for some $\lambda_i \geq 0$ and $t_i \in E^d$. Because $\dim(F_{i-1} \cap F_i) \geq 1$ for each $j$, we then see that $\lambda_i = \lambda_j$ and $t_i = t$. Hence, for any strongly connected family $\mathcal{F}$ of indecomposable faces of $P$, there are $\lambda \geq 0$ and $t \in E^d$ such that, if $G$ is the face of $Q$ corresponding to $F \in \mathcal{F}$, then $G = \lambda F + t$.

By the hypothesis of Theorem 2, there exists such a family $\mathcal{F}$ which touches every facet of $P$. If $F_i = P_{u_i}$ is such a facet, then there is a vertex $p$ of $F_i$ in some face of $\mathcal{F}$. So, if $q$ is the corresponding vertex of $g_i = Q_{u_i}$, we have $q = \lambda p + t$, and so the support parameters $\eta_i = \langle p, u_i \rangle$ and $\zeta_i = \langle q, u_i \rangle$ satisfy

$$\zeta_i = \langle q, u_i \rangle = \langle \lambda p + t, u_i \rangle = \lambda \eta_i + \langle t, u_i \rangle.$$  

We conclude that $Q = \lambda P + t$, as we wished to show.

3. Example and remark

We have just one example here to demonstrate the greater efficacy of our new criterion. If $P$ and $Q$ are two polytopes in $E^d$, we call the convex