ELEMENTS OF REDUCED TRACE 0

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ABSTRACT

Every element r of reduced trace 0 in a simple finite dimensional algebra R is a sum of at most 2 commutators. If R is not a division ring then r is a commutator, unless r is a scalar (in which case char(R) ≠ 0). The method of proof provides a generic division algebra of transcendence degree n² - 1.

Introduction

Throughout this paper R is a finite dimensional central simple algebra, with center F. Then R ⊗₁ F ≃ Mₙ(F) where F is the algebraic closure of F, and the reduced trace of an element r in R is defined as the trace of the matrix corresponding to r ⊗ 1 in Mₙ(F). Thus for any commutator r = [a, b] = ab - ba we have tr(r) = 0, and we address the converse question

QUESTION 1: If tr(r) = 0 then is r a commutator?

This is obvious for R = F (since then tr(r) = r), and is also true for R = Mₙ(F), cf. [6],[2]. Although the answer is unknown in general, there are various positive results, including for n = 2, 3 (Theorem 0.10). Also it turns out in general that r is a sum of at most two commutators, and we prove a slightly stronger fact. Actually there are two proofs, one of which involves Brauer factor

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sets (§3) and yields considerable extra information about generic matrix algebras, which we include as an appendix at the end.

We cannot yet answer Question 1 for a division algebra $D$, and its general answer seems to rely on properties of quadratic forms. We do have an affirmative answer for $R = M_n(D)$ whenever $n > 1$, unless the matrix is scalar and $n$ is prime to the characteristic of $F$. The proof is rather intricate, utilizing the other results of this paper, so we give a weakened result first (Theorem 1.10) and then the full result in section 2. The case where the matrix is scalar but $n$ is prime to $p = \deg D = \text{char}(F)$ is particularly intransigent, and is discussed separately.

The results of section 1 contain some facts concerning normal forms of matrices which might be of independent interest.

0. Some easy special cases

Remark 0.1: If $ac = ca$ then $[a, bc] = [a, b]c$ and $[a, cb] = c[a, b]$.

Remark 0.2: The difference of any element $b$ and any conjugate of $b$ is a commutator. Indeed

$$aba^{-1} - b = [a, b]a^{-1} = [a, ba^{-1}].$$

Conversely, writing $b = ca^{-1}$ we see $[a, b] = aca^{-1} - c$. Thus every commutator is a difference of conjugates, so Question 1 is equivalent to: if $\text{tr}(r) = 0$ then is $r$ a difference of two conjugates? Viewed in this way the question has a rather easy answer in several special cases.

PROPOSITION 0.3: Suppose $K/F$ is a cyclic field extension with $K \subset R$. Then any element $r$ of $K$ having trace 0 (with respect to the field extension) is a commutator in $R$.

Proof: Write $r = \sigma(b) - b$ for suitable $b$ in $K$, by the additive form of Hilbert's theorem 90. Then there is invertible $a$ in $R$ such that $\sigma(b) = aba^{-1}$, so

$$r = \sigma(b) - b = aba^{-1} - b = [a, ba^{-1}]$$

by Remark 0.2. □

Note: If $\alpha = \text{tr}_{K/F} r$ then $\text{tr} r = \frac{n}{[K:F]} \alpha$; thus if $K$ is a maximal subfield or more generally if $\text{char}(F) \nmid [K:F]$ then $\text{tr}_{K/F} r = 0$ is equivalent to $\text{tr} r = 0$. On the other hand, we have