LOCAL DIMENSION-FREE ESTIMATES FOR VOLUMES OF SUBLEVEL SETS OF ANALYTIC FUNCTIONS

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ABSTRACT

We derive sufficiently sharp local dimension-free estimates for volumes of sublevel sets of analytic functions in the unit ball of $\mathbb{C}^n$.

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1. Introduction and result

Let $F$ be a non-constant real-analytic function in the unit ball in $\mathbb{R}^n$. We are interested here in dimensionless estimates of the size of sub- and super-level sets \( \{x : |F(x)| \leq t\} \). To simplify the problem and avoid dependence on the domain of analyticity of $F$, we assume that $F$ is analytic in the complex unit ball in $\mathbb{C}^n$.

We denote complex balls \( \{z \in \mathbb{C}^n : |z - w| < r\} \) by $B_c(w, r)$, and real balls \( \{x \in \mathbb{R}^n : |x - u| < r\} \) by $B(u, r)$. For any real ball $B$, we denote by $\text{Vol}_B$ the normalized volume

\[
\text{Vol}_B(E) = \frac{\text{Vol}(B \cap E)}{\text{Vol}(B)}.
\]

Let $F$ be a non-constant analytic function in $B_c(0, 1)$, and let $B \subset B(0, 1 - \varepsilon)$ be a real ball. We look for the upper bounds for the distribution functions $t \mapsto \text{Vol}_B\{|F| \leq tM_B(F)\}$ ($t < 1$), and $t \mapsto \text{Vol}_B\{|F| \geq tM_B(F)\}$ ($t > 1$). The quantity $M_B(F)$ normalizes the distribution function of $|F|$ in $B$. We would like to keep our estimates dimensionless and universal: their right-hand sides should depend on a global “degree” $d_F$ of $F$ in the complex unit ball, and on the distance $\varepsilon$ from the ball $B$ to the unit sphere, but should not depend on the number of variables $n$, and on the choice of the ball $B$.

The choice of the degree

\[
d_F = \log \sup_{B_c(0,1)} |F| / |F(0)|
\]

is suggested by the one-dimensional case when local bounds follow from the classical Cartan lemma [L]. A traditional statistical normalization of the distribution function uses the median, that is, a number $m_B(F)$ such that

\[
\text{Vol}_B\{|F| \geq m_B(F)\} = \frac{1}{2}.
\]

To make the constants simpler, we choose for the normalization the $e^{-1}$-quantile, that is, a number $M_B(F)$ such that

\[
\text{Vol}_B\{|F| \geq M_B(F)\} = 1/e.
\]

Since $|F|$ is a real-analytic function, the quantile $M_B(F)$ is uniquely defined if $|F|$ is a non-constant function in $B(0,1)$.

Our main result is

**Theorem:** Let $F$ be a non-constant analytic function in the unit ball $B_c(0,1)$, and let $B$ be any real ball contained in $B(0,1-\varepsilon)$, $\varepsilon \leq \frac{1}{4}$. Then, for every $\lambda > 1$,

\[
\text{Vol}_B\{|F| \leq (C\lambda)^{-\sigma} M_B(F)\} \leq 1/\lambda,
\]

(1.1)