Generalization of Levinson's Theorem for All Composite Particles in a Multichannel Scattering Problem (*).

R. C. Hwa

Lawrence Radiation Laboratory, University of California - Berkeley, Calif.

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Summary. — On the basis of analyticity and unitarity, Levinson’s theorem is generalized for a multichannel scattering problem so that bound-state poles and resonance poles are put on the same footing. The sheet structure of the Riemann surface is analysed, and the condition for poles in the unphysical sheets is derived. The generalized formula relates the total number of composite-particle poles to the behavior of the S-matrix elements along their left-hand cuts.

Introduction.

In a single-channel two-body scattering problem, Levinson’s theorem has been generalized (1) to include the resonance poles in the unphysical sheet. It was found that a relationship exists between the total number of composite-particle poles and the phase change of the S-matrix along the left-hand cut. In this note we consider further the generalization to the case of many channels each having only two particles.

1. — One-channel case.

For the sake of completeness we summarize here the derivation for the one-channel case. The hypotheses are that the partial-wave S-matrix, \( S_1(s) \), is a real analytic function in the \( s \) (energy squared) plane, cut on the real axis along the real positive axis.

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from the threshold $s_i$ to $-\infty$ and from $-\infty$ to $s_L$, $s_L < s_i$; in the physical region $S_i(s)$ satisfies the unitarity condition, and in the asymptotic region as $|s| \to \infty$, $S_i(s)$ tends to a constant. The continuation of $S_i(s)$ to the unphysical sheet through the unitarity cut leads to the function $S_i^*(s)$, which can be established to be $S_i^{-1}(s)$. Thus if $n_0$ and $n_\nu$ designate respectively the number of zeros and of poles of $S_i(s)$ on the physical sheet, then $n_0 + n_\nu$ is the total number of poles (elementary and composite particles) in the two-sheeted Riemann surface bounded by the left-hand cut from $-\infty$ to $s_L$. On both sheets.

Consider the integral $I = \int ds S'(s)/S(s)$, where $S'(s)$ is the first derivative of $S(s)$ (the subscript $l$ having been suppressed) and $C$ is the largest possible closed counterclockwise contour in the cut $s$-plane. Thus $C$ consists of four parts: $C_L$, a contour tightly around the left-hand cut in a clockwise direction; $C_R$, tightly around the right-hand cut; and two large semicircles in the upper and lower half-planes. By Cauchy's theorem, $I$ is identically $(n_0 - n_\nu)2\pi i$. Since $S(s)$ tends to a constant asymptotically, the integrations along the semicircles contribute nothing. The contribution from $C_R$ is $2i \text{Im} \ln S(s)|_{s_L} = -4i[\delta(\infty) - \delta(s_L)]$, where $\delta$ is the (real) phase shift. On the basis of analyticity and unitarity, the usual Levinson theorem can be derived (2):

$$\delta(s_L) - \delta(\infty) = (n_\nu - n_\nu)\pi,$$

where $n_\nu$ is the number of elementary particles or CDD poles. Hence, we obtain $\ln S(s)|_{s_L} = (n_0 + n_\nu - 2n_\nu)2\pi i$, where the left-hand side implies the change $\ln S(s)$ undergoes as $s$ is taken along $C_L$. Since there is a zero of $S(s)$ associated with each elementary-particle pole—a property that can be made evident by reducing the strength of interaction between the elementary particle and the scattering system, whereupon the zero approaches the pole position—we have the formula $n_0 + n_\nu = 2n_\nu + n_\nu$; here $n_\nu$ is the total number of composite-particle poles that are on both sheets. We thus obtain

$$\ln S(s)|_{s_L} = 2\pi i n_\nu.$$

The meaning of this equation and its application to specific problems in giving an upper bound of $n_\nu$ are discussed in reference (1).

2. Sheet structure in the multichannel problem.

When there are $n$ coupled two-particle channels, we consider the Riemann surface consisting of all the sheets connected by the unitarity cut with $n$ normal thresholds. We derive here the structure of this surface and the total number of sheets. Let $A_{\nu}(s)$ be the partial-wave scattering amplitude from...

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