FORMS AND BAER ORDERED *-FIELDS

BY

KA HIN LEUNG

Department of Mathematics, National University of Singapore, Singapore, 119260
e-mail: matlhk@nus.edu.sg

ABSTRACT

It is well known that for a quaternion algebra, the anisotropy of its norm form determines if the quaternion algebra is a division algebra. In case of biquaternion algebra, the anisotropy of the associated Albert form (as defined in [LLT]) determines if the biquaternion algebra is a division ring. In these situations, the norm forms and the Albert forms are quadratic forms over the center of the quaternion algebras; and they are strongly related to the algebraic structure of the algebras. As it turns out, there is a natural way to associate a tensor product of quaternion algebras with a form such that when the involution is orthogonal, the algebra is a Baer ordered *-field iff the associated form is anisotropic.

1. Introduction

Let $D$ be a *-field, i.e. a division ring with an involution *. In $D$, we denote the set of nonzero symmetric elements by $S(D, *)$. A subset $P$ in $S(D, *)$ is called a Baer ordering if (i) $P + P \subseteq P$, (ii) $1 \in P$ and for any nonzero $x \in D$, $xPx^* \subseteq P$, (iii) $P \cup (-P) = S(D, *)$. In the literature, there are other types of orderings defined over *-fields; for a reference, see [C2].

Let $F$ be the center of $D$ and $F'$ be the fixed field of * in $F$. $(D, *)$ is called trivial if $D = F$ or $(D, *)$ is a standard quaternion algebra. Suppose $(D, *)$ is trivial. If $(D, *)$ admits a Baer ordering $P$, then $T' = \{ x_t x_t^*: x_t \in D \}$ is a preordering on $F'$. Conversely, if $T'$ is a preordering on $F'$, then as pointed out in [L2, Chapter 14], a $T'$-normed semiordering (as defined in [L2, Definition 14.4]) exists. It is clear from the definition of Baer ordering that any normed $T'$-semiordering on $F'$ is a Baer ordering on $(D, *)$. Let $T = \{ x_t x_t^*: x_t \in F \}$. When $D = F$, $T' = T$. Hence $(F, *)$ admits a Baer ordering iff $T$ is a preordering on $F'$. When $D = \left( \frac{a, b}{F} \right)$ and * is the standard involution, $T' = T + T(-a) +$
$T(-b) + T(ab)$. Hence, $T'$ is a preordering iff $T$ is a preordering and the form $(1, -a, -b, ab)$ is $T$-anisotropic. Thus, when $T$ is a preordering, the anisotropy of the $T$-form $(1, -a, -b, ab)$ implies the orderability of $(D, \ast)$. What about the case when $D = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ but $\ast$ is not standard? For $(D, \ast)$ to admit a Baer ordering, it is still necessary that $T$ is a preordering. In view of the earlier observation, we may ask the following question.

Does there exist a $T$-form $\phi(D, \ast)$ over $F'$ such that $(D, \ast)$ is Baer ordered iff $\phi(D, \ast)$ is anisotropic over $T$?

As we will see, the answer is affirmative when $D$ is a quaternion algebra. Moreover, such a result can be extended to the case when $D$ is a tensor product of quaternion algebras with $\ast$ satisfying certain conditions. Note that up until now, there is no easy way to determine if $(D, \ast)$ admits a Baer ordering even when $(D, \ast)$ is a quaternion algebra with an orthogonal involution. In [Le2], it is shown that a $\ast$-field $(D, \ast)$ admits a Baer ordering iff $(D, \ast)$ is Baer formally real. However, it is not easy to determine if $(D, \ast)$ is Baer formally real in general.

Our investigation on the orderability of quaternion algebras is also motivated by the following longstanding problem raised by Holland [H1]. Does every formally real $\ast$-field admit a Baer ordering? A $\ast$-field $(D, \ast)$ is said to be formally real if $\sum \alpha_i x \alpha_i^\ast \neq 0$ for any nonzero elements $\alpha_i$'s in $D$ and $x \in S(D, \ast)$. By using the results mentioned earlier, we see that the answer is affirmative when $(D, \ast)$ is trivial. Therefore, the next case to be considered is a quaternion algebra with a nonstandard involution. Thus, it is important to find a necessary and sufficient condition for a quaternion algebra to admit a Baer ordering.

2. Notation and preliminary results

From now on, we fix the following notation. $(D, \ast)$ is a $\ast$-field with center $F$ and $[D : F]$ is finite. For any subset $E$ in $D$, we denote $E \setminus \{0\}$ by $\hat{E}$.

To deal with noncommutative $\ast$-fields, we often make use of $\ast$-valuations. The notion was first introduced by Holland [H2]. The main purpose then was to lift Baer $\ast$-orderings from the residue $\ast$-fields, see [H2, C1]. In the papers [Le1, Le2], $\ast$-valuations are used to study $\ast$-fields finite dimensional over their centers as the dimension of the residue $\ast$-fields over their centers are usually smaller, and that allows us to apply an induction argument.

A valuation $v$ is said to be a $\ast$-valuation if $v(a) = v(a^\ast)$ for all $a \in \hat{D}$. As usual, we denote the valuation ring, residue class division ring and value group by $R_v, D_v$ and $\Gamma_D$ respectively. For any element $a$ in $R_v$, we denote its image in $D_v$ by $\hat{a}$. If $E$ is any subset of $D$, we denote $\{\hat{a} : a \in E \cap R_v\}$ by $\hat{E}$ and $v(\hat{E})$