COMPONENT GROUP OF THE $p$-NEW SUBVARIETY OF $J_0(Mp)$

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ABSTRACT

For an abelian variety $A$ over $\mathbb{Q}_p$, the special fibre in the Néron model of $A$ over $\mathbb{Z}_p$ is the extension of a finite group scheme over $\mathbb{F}_p$, called the group of connected components, by the connected component of identity. When $A$ is the Jacobian variety of an algebraic curve, its component group has been calculated in many cases. We determine in this paper the component group of the $p$-new subvariety of $J_0(Mp)$, for $M > 1$ a positive integer and $p \geq 5$ a prime not dividing $M$. Such a subvariety is not the Jacobian of any obvious curve, but it is not clear if it can ever be realised as the Jacobian of a curve.

1. Introduction

Let $A$ be an abelian variety over $\mathbb{Q}_p$ and let $A_{F_p}$ denote the special fibre of the Néron model of $A$ over $\mathbb{Z}_p$. Let $A_{F_p}^0$ denote the connected component of identity in $A_{F_p}$. The quotient $\Phi(A)_p \overset{\text{def}}{=} A_{F_p}/A_{F_p}^0$ is a finite group scheme that is étale over $\mathbb{F}_p$. This quotient is the group of connected components (or component group) of $A_{F_p}$.

When $A$ is the Jacobian variety of an algebraic curve, the component group of $A$ can be, and has been, computed in many cases (see, for example, [2], [10], [3], [9], [7]). However, the methods used in these cases rely heavily on the fact that $A$ is a Jacobian variety. When it is no longer obvious that $A$ is the Jacobian variety of a curve, these known methods cannot be extended and not much is known about the component group of such an abelian variety. In this paper,

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we calculate the component group of some abelian varieties that do not arise naturally as Jacobians of curves and on which the natural polarisation is not principal. However, we are not certain if the abelian varieties considered can be realised as the Jacobians of some curves or if they may admit some principal polarisation.

Let $M > 1$ be a positive integer and let $p > 5$ be a prime that does not divide $M$. Let $J_0(Mp)$ be the Jacobian variety, considered as an abelian variety over $\mathbb{Q}_p$, of the modular curve $X_0(Mp)$. There are two natural degeneracy maps $v_1, v_p: X_0(Mp) \to X_0(M)$. They induce, via Pic functoriality, two maps $v_1^*, v_p^*: J_0(M) \to J_0(Mp)$ on the Jacobian varieties. They also induce, via Albanese functoriality, two maps $(v_1)_*, (v_p)_*: J_0(Mp) \to J_0(M)$. The $p$-new subvariety $B$ of $J_0(Mp)$ is, by definition, the identity component of the intersection of the kernels of $(v_1)_*$ and $(v_p)_*$. Let $\Phi(B)_p$ denote the component group of the special fibre of the Néron model of $B$ over $\mathbb{Z}_p$. We determine $\Phi(B)_p$ and give an application of this knowledge.

**Theorem 1:** If $M > 1$ is a positive integer and $p \geq 5$ is a prime not dividing $M$, then there is a natural short exact sequence

\[
0 \to N \to \Phi(B)_p \to \Phi_{Mp,p} \to 0,
\]

compatible with the action of Hecke operators $T_n$ for $n \neq p$, where $\Phi_{Mp,p}$ is the component group of the special fibre of the Néron model of $J_0(Mp)$ over $\mathbb{Z}_p$ and the prime-to-$p$ part of $N$ is

\[
N(p) = \ker \left( J_0(M)(\mathbb{F}_{p^2})^{(p)} \to \hom_{\mathbb{F}_p}(\Sigma(M)(\mathbb{F}_p)(p), G_m) \right),
\]

where $\theta$ is a surjective map to be defined in §2, $J_0(M)(\mathbb{F}_{p^2})$ is the group of $\mathbb{F}_{p^2}$-rational points of the abelian variety $J_0(M)$ over $\overline{\mathbb{F}}_p$ and $\Sigma(M)$ is the Shimura subgroup of $J_0(M)$. Moreover, when the $p$-primary part of $J_0(M)(\mathbb{F}_{p^2})$ is trivial, (2) may be refined to yield

\[
N = \ker \left( J_0(M)(\mathbb{F}_{p^2}) \to \hom_{\mathbb{F}_p}(\Sigma(M)(\mathbb{F}_p)(p), G_m) \right).
\]

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