ON TRIANGULAR SUBALGEBRAS OF GROUPOID C*-ALGEBRAS

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ABSTRACT
Let \( \mathfrak{B} \) be an AF C*-algebra with Stratila-Voiculescu masa \( \mathfrak{D} \) and let \( \mathfrak{A} \) be a maximal triangular subalgebra of \( \mathfrak{B} \) with diagonal \( \mathfrak{D} \). Peters, Poon and Wagner showed that \( \mathfrak{A} \) need not be a C*-subdiagonal subalgebra of \( \mathfrak{B} \) in the sense of Kawamura and Tomiyama. We investigate and explain this phenomena here from the perspective of groupoid C*-algebras by representing \( \mathfrak{A} \) as the "incidence algebra" associated with a topological partial order. A number of examples are given showing what can keep a maximal triangular algebra from being C*-subdiagonal.

§1. Introduction

Our objective in this note is to combine the analysis of [MS] with [PPW] and [T] to help clarify the structure of maximal triangular subalgebras of AF C*-algebras and to resolve an issue that arose in the writing of [MS].

We follow the notation and terminology of [R] and [MS] with regard to groupoids. Suppose that \( G \) is a 2nd countable, locally compact, \( r \)-discrete, amenable, principal groupoid admitting a Haar system and a cover by compact open \( G \)-sets, suppose \( \sigma \) is a two cocycle on \( G \) with values in \( T \) and let \( C^*(G, \sigma) \) be the associated C*-algebra. In [MS], we show that given a norm closed subalgebra \( \mathfrak{A} \) of \( C^*(G) \) such that \( \mathfrak{A} \cap \mathfrak{A}^* = C^*(G^0) \) (such an algebra is called triangular), then there is an open subset \( P \subseteq G \) such that \( P \cdot P \subseteq P \) and \( P \cap P^{-1} = G^0 \) and such that \( \mathfrak{A} \) is the closure in \( C^*(G) \) of the set of all functions in \( C_c(G) \) that are supported in \( P \). We write \( \mathfrak{A} = \mathfrak{A}(P) \). The correspondence between triangular algebras and open sets \( P \) with the indicated properties is one to one, so the notation is justified. We note that our terminology differs slightly from that of Peters, Poon and

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Wagner. Their triangular algebras need not be closed. This difference will be of no consequence here.

Groupoids of the kind we are considering may be viewed as equivalence relations on $G^0$, i.e., certain subsets of $G^0 \times G^0$, but with topologies that may be finer than the product topology. The sets $P$, then, are (the graphs of) partial orders on $G^0$ and the algebras $\mathfrak{A}(P)$ may be viewed as one possible generalization of incidence algebras.

A triangular algebra $\mathfrak{A}(P)$ is called **maximal** triangular if it is not contained in any larger triangular subalgebra. In terms of the set $P$, this means that $P$ is not contained in any larger partial order on $G^0$. If $G^0$ were finite, then this would be the case if and only if $P \cup P^{-1} = G$, i.e., if and only if $P$ totally orders each equivalence class determined by $G$. When [MS] was written, we did not know if a maximal partial order has this property in general. This was important for [MS] because an algebra $\mathfrak{A}(P)$, with $P$ satisfying $P \cup P^{-1} = G$, has additional structures that makes it tractable for analysis. Indeed, such algebras turn out to be what are known as maximal $C^*$-subdiagonal algebras [KT] and share many structural properties with the full algebra of upper triangular matrices. One of the consequences of [PPW] is that maximal triangular algebras $\mathfrak{A}(P)$ do exist for which $P$ fails to satisfy $P \cup P^{-1} = G$. Moreover, examples may be found in the context of AF $C^*$-algebras. Recall [R, II.1.15] that every AF $C^*$-algebra may be realized in terms of an AF groupoid (and in that case $\sigma = 1$). Analysis of their examples as well as the examples in [T] show that the sets $P$ constructed are open, but not closed. However, if $P$ is open and $P \cup P^{-1} = G$, then $P$ is closed because $P = G^0 \cup G \setminus P^{-1}$. This and Proposition 1.1 lead one to speculate that if $P$ corresponds to a maximal triangular algebra and if $P$ is closed, then $P \cup P^{-1} = G$. The main example of this note, Example 3.7, shows that this is not the case. Its construction is rather complicated and its existence is rather surprising. The latter calls for an effective criterion for deciding when an open partial order in an AF groupoid is closed. This is the goal of Theorem 1.3, below, but before giving it we present Proposition 1.1 giving a usable sufficient condition implying $P \cup P^{-1} = G$ when $\mathfrak{A}(P)$ is maximal. For it, recall that $\mathfrak{A}(P)$ is called a **nest algebra** if there is a totally ordered family, $\{p_\alpha\}_{\alpha \in A}$, of projections in the multiplier algebra of $C^*(G^0)$ such that

$$\mathfrak{A}(P) = \{a \in C^*(G, \sigma) | (1 - p_\alpha)ap_\alpha = 0 \text{ for all } \alpha \in A\}.$$ 

**Proposition 1.1.** If $\mathfrak{A}(P)$ is a nest subalgebra of $C^*(G, \sigma)$ and if $P$ is closed, then $P \cup P^{-1} = G$. That is, $\mathfrak{A}(P)$ is a maximal $C^*$-subdiagonal subalgebra of $C^*(G, \sigma)$. 
