ORDINARY \((2m + 1)\)-POLYTOPES

BY

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ABSTRACT

For each \(k, m\) and \(n\) such that \(n \geq k \geq 2m + 1 \geq 5\), we present a convex \((2m + 1)\)-polytope with \(n + 1\) vertices and \(2\binom{k-n}{m} + \binom{n-k}{m-1} 2\) facets with the property that there is a complete description of each of the facets based upon a total ordering of the vertices.

Introduction

We introduce a class of convex \((2m + 1)\)-polytopes \(P\), via a total ordering of the vertices of \(P\), which contains the cyclic \((2m + 1)\)-polytopes and which has the property that there is a complete description of the facets of each \(P\). These polytopes, which we call ordinary, have been defined for \(m = 1\) in [1] and we present them here for \(m > 1\). In fact, we define an ordinary \(d\)-polytope for any \(d \geq 3\) but show that the polytope is not cyclic only if \(d = 2m + 1\) (Theorem A).

As guide-posts, we indicate the central concepts and results of our theory.

Let \(P\) be a convex \(d\)-polytope in \(E^d\), \(d = 2m + 1 \geq 5\), with a totally ordered set of vertices, say, \(x_0 < x_1 < \cdots < x_n\). Then \(P\) is ordinary if each of its facets satisfies a global condition (the necessary part of Gale's Evenness Condition) and a local one (a specific relation among the vertices of a facet). Then there exist integers \(k\) and \(l\) (see Lemma 4 for the existence of \(k\)) such that \(d \leq k\), \(l \leq n\), \(\text{conv}\{x_0, x_i\}\) is an edge of \(P\) if and only if \(1 \leq i \leq k\), and \(\text{conv}\{x_{n-i}, x_n\}\) is an edge of \(P\) if and only if \(1 \leq i \leq l\). In fact, \(k\) is equal to \(l\) (Corollary 13) and we

Received July 11, 1994
call it the characteristic of $P$. Given $k$ and $l$, we list the facets of $P$ containing $x_0$ or $x_n$ in Lemmas 8 and 9, and the other facets of $P$ in Lemma 11. In Theorem B and its Corollary, we describe completely these facets and show that if $k$ is the characteristic of $P$ then

$$f_{2m}(P) = 2 \binom{k-m}{m} + (n-k) \binom{k-m-2}{m-1},$$

and that if $k = n$ then $P$ is cyclic.

Finally, we note that ordinary 3-polytopes were inspired by the idea of choosing, as vertices, points on a convex ordinary space curve in $E^3$. Unfortunately, there is as yet no definition of a convex ordinary space curve in $E^d$ for $d > 3$. However, certain types of curves in $E^d$ (for example, curves of order $d$) have properties that are independent of $d$, as long as the parity of $d$ is the same. Thus our expectation, in generalizing the definition of an ordinary 3-polytope, is that there is a new class of $d$-polytopes only if $d = 2m + 1$. As this is the case, our approach seems to be a reasonable one.

1. Definitions

Let $Y$ be a set of points in $E^d$, $d \geq 3$. Then $\text{conv} Y$ is the convex hull of $Y$ and if $Y = \{y_1, \ldots, y_s\}$ is finite, we set

$$[y_1, \ldots, y_s] = \text{conv}\{y_1, \ldots, y_s\}.$$

Thus, $[y_1, y_2]$ is the closed segment with end points $y_1$ and $y_2$.

Let $V = \{x_0, x_1, \ldots, x_n\}$ be a totally ordered set of $n + 1$ points in $E^d$ with $x_i < x_j$ if and only if $i < j$. We say that $x_i$ and $x_{i+1}$ are successive points, and if $x_i < x_j < x_k$ then $x_j$ separates $x_i$ and $x_k$ or $x_j$ is between $x_i$ and $x_k$.

Let $Y \subset V$. Then $Y$ is connected (in $V$) if $x_i < x_j < x_k$ and $\{x_i, x_k\} \subset Y$ imply that $x_j \in Y$. If $Y$ is not connected then clearly it can be written uniquely as the union of maximal connected subsets, which we call components of $Y$.

A component $X$ of $Y$ is even or odd according to the parity of $|X| = \text{card} X$. Next, $Y$ is a Gale set (in $V$) if any two points of $V \setminus Y$ are separated by an even number of points of $Y$. Finally, $Y$ is a paired set if it is the union of mutually disjoint subsets $\{x_i, x_{i+1}\}$.

We note that $V$, $\emptyset$ and all paired subsets of $V$ are Gale sets. Conversely, let $Y \subset V$ be a Gale set. If $Y \cap \{x_0, x_n\} = \emptyset$ then $Y$ is a paired set. Thus if $Y$ is