METRICS ON PRODUCTS OF SURFACES WITH NON-POSITIVE SECTIONAL CURVATURE

BY

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ABSTRACT

Let \((S_1, g_1), i = 1, 2\) be two compact riemannian surfaces isometrically embedded in euclidean spaces. In this paper we show that if \(M = S_1 \times S_2\), then for any function \(F: M \rightarrow \mathbb{R}\), the graph of \(F\), i.e. the manifold \(\{(x, F(x)) : x \in M\}\), does not have positive sectional curvature.

1. Introduction

Let \(M\) be a riemannian manifold and let \(T_pM\) denote the tangent vector space of \(M\) at \(p\). The sectional curvature is the function that assigns the Gauss curvature at \(p\) of the surface built of geodesics starting at \(p\) and velocity vector in \(\sigma\) to any 2-dimensional space \(\sigma \subset T_pM\). We say that the riemannian manifold \(M\) has positive sectional curvature if for every point \(p \in M\) the sectional curvature \(K(\sigma)\) of every 2-plane \(\sigma \subset T_pM\) is positive. An example of such manifolds are the \(n\)-dimensional spheres of radius \(r\), \(S^n(r)\), with the metric induced by \(\mathbb{R}^{n+1}\). In this case its sectional curvature is equal to \(1/r^2\) for any 2-plane \(\sigma\) in \(T_pM\). In general, the question of deciding if a given manifold admits a riemannian metric with positive sectional curvature is a difficult one; for example, the conjecture stating that no riemannian metric on \(S^2 \times S^2\) has positive sectional curvature is known as Hopf's conjecture and remains unsolved. In this paper, we prove that a certain type of metric on a product of surfaces cannot have positive sectional curvature.

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2. Main theorem

In this section we state and prove the main theorem of this paper.

**Theorem 2.1:** Let $S_i \subset \mathbb{R}^{k_i}$, $i = 1, 2$ be two compact riemannian surfaces with the metric induced by the euclidean spaces. If $M = S_1 \times S_2$, in particular $M \subset \mathbb{R}^N$ with $N = k_1 + k_2$, then for any smooth function $F: M \rightarrow \mathbb{R}$, the manifold $\tilde{M} = \{(x, F(x)) \in \mathbb{R}^{N+1} : x \in M\}$ with the metric induced by $\mathbb{R}^{N+1}$ does not have positive sectional curvature.

**Remark:** A generalization of this theorem to functions with values in $\mathbb{R}^k$ will provide a proof of Hopf’s conjecture.

Before proving this theorem, we fix some notation and prove some lemmas that will help us to relate the sectional curvature on $M$ and $\tilde{M}$. Let us denote by $\tilde{\nabla}$ the connection on $M$ with the metric induced by the embedding $\phi(x) = (x, F(x))$ of $M$ in $\mathbb{R}^{N+1}$, and let us denote by $\nabla$ the connection on $M$ induced by $\mathbb{R}^N$. We will use the following notation.

1. If $m \in M$, we denote by $\tilde{m}$ the point $(m, F(m))$. We denote by $\tilde{M}$ the manifold $\tilde{\phi}(M) \subset \mathbb{R}^{N+1}$ with the metric induced by $\mathbb{R}^{N+1}$.

2. If a map $Y: M \rightarrow \mathbb{R}^N$ defines a tangent vector field on $M$, we denote by $\tilde{Y}$ the vector field on $\tilde{M}$ defined by $\tilde{Y}(\tilde{m}) = (Y(m), dF_m(Y(m)))$.

3. Given $p \in M$ and $v \in \mathbb{R}^N$, we denote by $v_T(p)$ the tangent orthogonal projection of $v$ on $T_pM$. Given $w \in \mathbb{R}^{N+1}$, we denote by $w_T(p)$ the tangent orthogonal projection of $w$ on $T_p\tilde{M}$.

Since the manifold $\tilde{M}$ is isometric to the manifold $M$ with the metric induced by the embedding $\phi(p) = (p, F(p))$, then we also denote by $\tilde{\nabla}$ the connection on $\tilde{M}$. We will find the sectional curvature on $\tilde{M}$ in terms of the sectional curvature of $M$ and the derivatives of $F$. For any $m \in M \subset \mathbb{R}^N$, let $\{v_i : i = 1, \ldots, n\}$ be an orthonormal frame defined in an open neighborhood $U \subset M$ of $m$; note that each $v_i: U \rightarrow \mathbb{R}^N$ is a tangent vector field. Without loss of generality we may assume that the vector fields $\nabla v_i v_j$ vanish at $m$ for all $i, j$. We denote by $\nabla F$ the gradient vector of $F$ as a function on $M$; since the frame of the vector field $v_i$’s is orthonormal, then for any $p \in U$ we have that $\nabla F = \sum_{i=1}^n dF_p(v_i(p))v_i(p)$. Recall that the hessian of $F$ is the symmetric 2-tensor given by $\text{Hess}(F)(X, Y) = \langle \nabla_X(\nabla F), Y \rangle$ for any pair of tangent vector fields on $M$. For any $p \in U$, we define $F_i(p) = dF_p(v_i(p))$ and $F_{ij}(p) = (\text{Hess}(F))_{p}(v_i(p), v_j(p))$. Before trying to find a relation between the sectional curvature of $M$ and $\tilde{M}$, we need to prove the following lemmas,