THE STOCHASTIC ACCELERATION PROBLEM
IN TWO DIMENSIONS

BY

TOMASZ KOMOROWSKI
Institute of Mathematics, UMCS
pl. Marii Curie Sklodowskiej 1, 20-031 Lublin, Poland
e-mail: komorow@hektor.umcs.lublin.pl

AND

LENYA RYZHIK
Department of Mathematics, University of Chicago
Chicago, IL 60637, USA
e-mail: ryzhik@math.uchicago.edu

ABSTRACT
We consider the motion of a particle in a two-dimensional spatially homogeneous mixing potential and show that its momentum converges to the Brownian motion on a circle. This complements the limit theorem of Kesten and Papanicolaou (1980) proved in dimensions $d \geq 3$.

1. Introduction
The momentum of a particle moving in a weakly random Hamiltonian field approaches in the long time limit the Brownian motion on the level set of the Hamiltonian in the momentum space. The position of the particle follows the trajectory generated by this momentum process. This limit has been first investigated rigorously by Kesten and Papanicolaou in [4] in dimension $d \geq 3$. More precisely, they have considered a Hamiltonian of the form

$$\mathcal{H}(x, v) = \frac{v^2}{2} + \sqrt{\delta}H(x), \quad x \in \mathbb{R}^d, v \in \mathbb{R}^d, v = |v|$$

Received June 14, 2005
with a spatially homogeneous and mixing random field $H(x)$, and $0 < \delta \ll 1$. The corresponding particle trajectories are
\[
\frac{dX}{dt} = V, \quad \frac{dV}{dt} = -\sqrt{\delta} \nabla H(X), \quad X(0) = 0, V(0) = v_0.
\]
As the random potential is weak, its effect becomes appreciable over large times — of the order $T \sim O(1/\delta)$. Accordingly, we introduce the re-scaled process $X^\delta(t) = \delta X(t/\delta)$, $V^\delta(t) = V(t/\delta)$ that satisfies
\[
\frac{dX^\delta}{dt} = V^\delta, \quad \frac{dV^\delta}{dt} = -\frac{1}{\sqrt{\delta}} \nabla H\left(\frac{X^\delta}{\delta}\right), \quad X^\delta(0) = 0, V^\delta(0) = v_0.
\]
Kesten and Papanicolaou have shown that the process $V^\delta(t)$ converges in law as $\delta \downarrow 0$ to a Brownian motion $V(t)$ on the sphere $S^{d-1} = \{|V| = v_0\} \subset \mathbb{R}^d$. The process $X^\delta(t)$ converges (also in law) to $X(t)$, the time integral of $V(t)$:
\[
X(t) = \int_0^t V(s)ds.
\]
Later, Dürr, Goldstein and Lebowitz have considered the two-dimensional case [2] with a potential $H(x)$ of the form
\[
H(x) = \sum_j V(x - r_j).
\]
Here $r_j$ are the locations of randomly distributed Poisson scatterers and $V$ is a compactly supported sufficiently smooth potential. They used a martingale technique to establish a result similar to that of Kesten and Papanicolaou in this case.

The goal of the present paper is to prove the diffusive limit in the general two-dimensional setting with the same assumptions on the random potential as in the original paper of Kesten and Papanicolaou. We recall that their proof was based on the following method. The main difficulty in obtaining the limit is that the random potential is time-independent, hence the time increments of $X^\delta(t)$ may be correlated: this happens when the trajectory comes close to its own past. To handle this issue one modifies the trajectories of the Hamiltonian system in such a way that the modified system has a better chance of being Markovian in time. The modification guarantees two properties: (i) the new trajectories will always go away from the regions of the physical space that they have just visited, and (ii) self-intersections do not lead to “gaining information about the past”. The former is achieved by keeping momenta aligned locally in