PROXIMAL FLOWS OF LIE GROUPS

BY
S. GLASNER

ABSTRACT
Recent results of M. Ratner enable us to solve an open problem on the existence of proximal (and not strongly proximal) minimal actions of Lie groups.

Let \((G, X)\) be a \(G\)-flow, i.e. \(X\) is compact Hausdorff and \(G\) a locally compact group acting on \(X\) \(((g, x) \mapsto gx\) is continuous and \(g \mapsto L_g\) is a homomorphism of \(G\) into the group of self-homeomorphisms of \(X\)). This action induces an action of \(G\) on the space \(\mathcal{M}(X)\) of probability measures on \(X\) endowed with the weak * topology, making \((G, \mathcal{M}(X))\) a \(G\)-flow. We say that \(Y \subset X\) is a minimal set or that \((G, Y)\) is a minimal \(G\)-flow if \(Gy = Y\) for every \(y \in Y\). \((G, X)\) is proximal if \(\Delta = \{(x, x) : x \in X\}\) contains all the minimal subsets of the flow \((G, X \times X)\) where \(g(x, y) = (gx, gy)\). \((G, X)\) is strongly proximal (s.p.) if \(\hat{X} = \{\delta_x : x \in X\}\), the collection of point masses, contains every minimal subset of \((G, \mathcal{M}(X))\).

For every group \(G\) there exists a unique, up to isomorphism, minimal proximal (s.p.) flow \((G, \Pi(G))\) \(((G, \Pi, (G)))\) which admits every minimal proximal (s.p.) \(G\)-flow as a factor, [2]. When \(G\) is solvable \(\Pi_i(G)\) is trivial and \(\Pi(G)\) is trivial for nilpotent \(G\). When \(G\) is a Lie group it was shown by Furstenberg that \(\Pi_i(G)\) is a \(G\)-homogeneous space and in particular for \(G\) semisimple with finite center \(\Pi_i(G) = G/P\) where \(P\) is the normalizer of \(N\) in \(G\) and \(G = KAN\) is an Iwasawa decomposition, [3]. The following question was posed in [2]: for a Lie group \(G\) is \((G, \Pi(G))\) isomorphic to \((G, \Pi, (G))\), in other words are there minimal proximal and non-strongly-proximal \(G\)-flows. We use recent profound results of M. Ratner on the horocycle flows to show that the action of the solvable group

\[
S = \left\{ \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} : a, t \in \mathbb{R}, a > 0 \right\}
\]

Received May 5, 1983

97
on the homogeneous space $G/\Gamma$ where $G = \text{SL}(2, \mathbb{R})$ and $\Gamma$ is a maximal discrete co-compact and non-arithmetic subgroup, is proximal. Since

$$S \supset N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

and $(N, G/\Gamma)$, the horocycle flow, is minimal, so is $(S, G/\Gamma)$. Since $S$ is solvable the latter flow can not be strongly proximal. (This can be seen directly since the Haar measure on $G/\Gamma$ is left invariant by $G$ and hence by $S$.) The same of course is true for the flow $(G, G/\Gamma)$. Thus we conclude that $\Pi(S)$ is not trivial and that $\Pi(G) \neq \Pi_0(G)$.

Further results of Ratner permit us to conclude also that for every uniform (i.e. discrete and co-compact) subgroup $\Gamma$ of $G$, there exists a maximal uniform non-arithmetic subgroup $\Gamma_2$ such that $(\Gamma, G/\Gamma_2)$ is minimal and proximal. In particular $\Pi(\Gamma) \neq \Pi_0(\Gamma)$ for every uniform subgroup $\Gamma$ of $\text{SL}(2, \mathbb{R})$.

We now proceed to show that $(S, G/\Gamma)$ is proximal when $\Gamma$ is a maximal uniform non-arithmetic subgroup of $G$. For $s \in \mathbb{R}$ let

$$g_s = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix},$$

then $g_s \in S$ and for

$$h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in N$$

we have $g_s h_t = h_t g_s$. Put $X = G/\Gamma$; our first step is to show that every minimal set in $(N, X \times X)$ is of the form $\Delta_t = \{(x, h_t x) : x \in X\}$ for some $t \in \mathbb{R}$. In fact if $M \subset X \times X$ is an $N$-minimal set then there exists an $N$-invariant probability measure $\mu$ on $M$, which by unique ergodicity of $(N, X)$, [1], is a self joining. By [4] th. 8.3 and cor. 8.2, there exists $t \in \mathbb{R}$ such that the set $\Delta_t$ has $\mu$-measure 1. Since both $\Delta_t$ and $M$ are minimal we have $M = \Delta_t$.

Next let $x, y \in X$; there is an $N$-minimal set contained in $\overline{N(x, y)}$ and by the preceding paragraph there exists $t \in \mathbb{R}$ with $\Delta_t \subset \overline{N(x, y)}$. If $t = 0$ we are done. If not then for $s \in \mathbb{R}$ we have $g_s \Delta_t = \Delta_{s+t}$ and hence $S(x, y) \supset \bigcup_{s > 0} \Delta_{s+t}$. By minimality of $(N, X)$ the latter set is dense in $X \times X$. In particular $S(x, y) \supset \Delta = \Delta_0$ and $x$ and $y$ are proximal. This completes the proof.

REMARKS. (1) Since we have shown that $(G, G/\Gamma)$ is proximal so is $(H, G/\Gamma)$ for every co-compact subgroup $H$ of $G$.

(2) If two strictly ergodic $N$-flows $(N, X)$ and $(N, Y)$ carry invariant measures $\mu_X$ and $\mu_Y$ respectively, with respect to which they are measure theoretically