BANACH SPACES EMBEDDING INTO $L_0$

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ABSTRACT

Our main result in this paper is that a Banach space $X$ embeds into $L_1$ if and only if $l_1(X)$ embeds into $L_0$; more generally if $1 \leq p < 2$, $X$ embeds into $L_p$ if and only if $l_p(X)$ embeds into $L_0$.

1. Introduction

It is still unknown whether every Banach space which embeds into $L_0 = L_0(0, 1)$ is isomorphic to a subspace of $L_1$. This problem was suggested by Kwapien [9]. Our main result in this paper is that a Banach space $X$ embeds into $L_1$ if and only if $l_1(X)$ embeds into $L_0$; more generally if $1 \leq p < 2$, $X$ embeds into $L_p$ if and only if $l_p(X)$ embeds into $L_0$.

Before discussing the problem and the contents of the paper, we introduce some notation. Throughout the paper $\Omega$ will denote a compact metric space, $\Sigma$ the $\sigma$-algebra of Borel subsets of $\Omega$ and $P$ a nonatomic probability measure on $\Sigma$. Of course there is no loss of generality in taking $\Omega = [0, 1]$ and $P$ Lebesgue measure on $[0, 1]$. For $0 \leq p < \infty$, $L_p(\Omega, \Sigma, P)$ will be abbreviated to $L_p$. We also denote by $L(p, \infty)$ the Lorentz space, weak $L_p$, of all $f \in L_0(\Omega, \Sigma, P)$ so that

$$\|f\|_{L_p} = \sup_{0 < x < \infty} x(P(|f| > x))^{1/p} < \infty.$$ 

Let us say that a linear operator $V : X \to L_p$ (or $V : X \to L(p, \infty)$) is a strong embedding if $V$ is an isomorphism onto its range, and the topology of $L_p$ (or $L(p, \infty)$) on its range coincides with the $L_0$-topology (convergence in measure).

For $1 \leq p \leq 2$, a Banach space $X$ is said to be of type $p$ (Rademacher) if there

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is a constant $C$ so that

$$\text{Avg} \left\| \sum_{i=1}^{n} e_i x_i \right\|^p \leq C \sum_{i=1}^{n} \| x_i \|^p$$

where the average is taken over all choices of sign $e_i = \pm 1$. Every Banach space is of type one.

The first progress on Kwapien's problem was made by Nikishin [14] who showed that a Banach subspace of $L_0$ is in fact isomorphic and is a subspace of $L_p$ for every $p < 1$. Later [15] (cf. expositions in [13] and [5]) he refined this result to establish the following factorization theorem:

**Theorem 1.1. (Nikishin).** Let $X$ be a Banach space of type $p$ ($1 \leq p < 2$) and let $V : X \to L_0$ be any continuous linear operator. Then given $\varepsilon > 0$, there exists a set $E$ with $P(E) \geq 1 - \varepsilon$ so that if

$$Wx = 1_{E}Vx$$

then $W$ is a bounded linear operator from $X$ into $L(p, \infty)$.

**Corollary 1.2.** Every Banach subspace of $L_0$ can be strongly embedded in $L(1, \infty)$.

**Corollary 1.3.** Every Banach subspace of $L_0$ of type $p$ can be strongly embedded in $L(p, \infty)$.

We obtain the results announced in the introduction by a close analysis of the spaces $L(p, \infty)$. Our methods hinge on the existence of non-trivial continuous linear functionals on the non-locally convex quasi-Banach space $L(1, \infty)$. This fact was first observed by Cwikel and Sagher [2] and recently the dual of $L(1, \infty)$ has been studied by Cwikel and Fefferman [1] and by Kupka and Peck [8]. Our methods are quite similar to techniques in [8] but were obtained independently.

In fact for convenience, in Section 2, we study not $L(p, \infty)$ but instead the $l_\infty$-product $l_\infty(L(p, \infty))$ which we abbreviate to $\mathcal{Y}_p$. Thus $\mathcal{Y}_p$ consists of all sequences $F = (f_n)$ where $f_n \in L(p, \infty)$ and

$$\| F \| = \sup_n \| f_n \|_{p, \infty} < \infty.$$ 

We do, however, give an application of these ideas to $L(p, \infty)$ showing the standard embedding of $L_p$ into $L(p, \infty)$ using $p$-stable processes yields a complemented subspace.