ON DOMINATED $l_1$ METRICS*

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ABSTRACT

We introduce and study a class $l_{1}^{\text{dom}}(\rho)$ of $l_1$-embeddable metrics corresponding to a given metric $\rho$. This class is defined as the set of all convex combinations of $\rho$-dominated line metrics. Such metrics were implicitly used before in several constructions of low-distortion embeddings into $l_p$-spaces, such as Bourgain’s embedding of an arbitrary metric $\rho$ on $n$ points with $O(\log n)$ distortion. Our main result is that the gap between the distortions of embedding of a finite metric $\rho$ of size $n$ into $l_2$ versus into $l_{1}^{\text{dom}}(\rho)$ is at most $O(\sqrt{\log n})$, and that this bound is essentially tight. A significant part of the paper is devoted to proving lower bounds on distortion of such embeddings. We also discuss some general properties and concrete examples.

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1. Introduction

Approximate embeddings of finite metric spaces into \( l_p \)-spaces were first studied mainly in connection with problems in Functional Analysis (Local Theory of Banach spaces; see, for instance, [6], [7], [14], [4]). Currently they constitute a rich research area with intimate links to Functional Analysis, to classical questions of Combinatorial Optimisation, and to the design of approximate algorithms. In the last area, it offered new paradigms which allowed one to solve long-standing open problems. A partial list of results and applications includes [16, 15, 2, 9, 18, 12].

Let \( (X, \rho) \) be a metric space. If \( f: X \to Z \) is an embedding of \( X \) into a normed space \( (Z, \| \cdot \|) \), we say that \( f \) has **distortion at most** \( D \) if we have

\[
\frac{1}{D} \| f(x) - f(y) \| \leq \rho(x, y) \leq D \| f(x) - f(y) \| \quad \text{for all } x, y \in X
\]

(that is, \( f \) is non-expanding and contracts each distance by the factor of at most \( D \)). For \( (X, \rho) \) finite, we define \( c_p(\rho) \) as the infimum of \( D \geq 1 \) such that there is an embedding with distortion at most \( D \) into \( l_p^m \) (for some natural number \( m \)). Here \( l_p^m \) denotes the space \( \mathbb{R}^m \) with the norm \( \| x \|_p = (\sum_{i=1}^{m} |x_i|^p)^{1/p} \). For \( (X, \rho) \) infinite, we define \( c_p(\rho) \) as the supremum of \( c_p(\tau) \), where \( \tau \) is \( \rho \) restricted to a finite subspace of \( X \) (in this way, we avoid discussion of infinite-dimensional \( l_p \)-spaces).

The most important cases, both theoretically and practically, are \( p = 1 \) and \( p = 2 \). Much of the existing theory deals with finding good bounds on \( c_1(\rho) \) and \( c_2(\rho) \), under various conditions on \( \rho \), and with constructing mappings achieving optimal or near-optimal distortions.

In this paper we introduce and study certain subclasses of \( l_1 \)-metrics. For each finite metric space \( (X, \rho) \), we define a class \( l_{1, \text{dom}}(\rho) \) of metrics on \( X \). Such metrics have been used implicitly before. They appear, e.g., in Bourgain’s proof [4] showing that each \( n \)-point metric \( \rho \) can be embedded into \( l_2 \) with distortion \( O(\log n) \), and in Rao’s [18] improvement of this construction for metrics of a special type (see Section 6 below for an example of such construction). The metrics in \( l_{1, \text{dom}}(\rho) \) have a relatively simple structure, yet many interesting properties, and a better understanding of such metrics may well prove rewarding.

Let \( (X, \rho) \) be a finite metric space. A **\( \rho \)-dominated line metric** on \( X \) is a metric \( \tau \) induced by a nonexpanding (1-Lipschitz) embedding of \( (X, \rho) \) into the real line. Explicitly, \( \tau(x, y) = |\phi(x) - \phi(y)| \leq \rho(x, y) \) for some nonexpanding mapping \( \phi: X \to \mathbb{R} \). The class \( l_{1, \text{dom}}(\rho) \) consists of all metrics on the ground set