QUADRATIC BASE CHANGE OF $\theta_{10}$

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ABSTRACT

In case of $GL_n$ over p-adic fields, it is known that Shintani base change is well behaved. However, things are not so simple for general reductive groups. In the first part of this paper, we present a counterexample to the existence of quadratic base change descent for some Galois invariant representations. These are representations of type $\theta_{10}$. In the second part, we compute the local $L$-factor of $\theta_{10}$. Unlike many other supercuspidal representations, we find that the $L$-factor of $\theta_{10}$ has two poles. Finally, we discuss these two results in relation to the local Langlands correspondence.

Introduction

Let $k_0$ be a $p$-adic field with odd residue characteristic and let $k$ be a cyclic Galois extension of $k_0$. Let $Gal(k/k_0)$ be its Galois group generated by $\sigma$. Let $G$ be a connected reductive algebraic group defined over $k_0$ and $G_{k_0}$ (resp. $G_k$) be its $k_0$-rational (resp. $k$-rational) points. Let $\hat{G}_{k_0}$ be the set of irreducible

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admissible representations $\pi$ of $G_{k_0}$ and let $\hat{G}_k^\sigma$ be the set of irreducible admissible representations $\Pi$ of $G_k$ which are $\sigma$-invariant, that is, $\Pi \simeq \Pi \circ \sigma$.

In general, the conjectural Shintani lifting describes a (surjective) map from $\hat{G}_{k_0} / \sim$ to $\hat{G}_k^\sigma$ defined via a twisted character formula where for $\pi, \pi' \in \hat{G}_{k_0}$, $\pi \sim \pi'$ if and only if $\pi \simeq \pi' \otimes \chi$ for a character $\chi$ of $k_0^\times$ which is trivial on the image of the norm map $N_{k/k_0}$. More precisely, this map can be defined as follows:

**Definition [AC, La]**: Let $\pi$ and $\Pi$ be irreducible, admissible representations of $G_{k_0}$ and $G_k$ respectively. Suppose that $\Pi$ is Galois invariant. Then we can extend $\Pi$ to a representation of the semi-direct product $G_k \rtimes \langle \sigma \rangle$. We say that $\Pi$ is a (base change) lift or Shintani ascent of $\pi$ if for any $g \in G_k$ such that $N_{k/k_0}(g)$ is regular and for some extended representation $\tilde{\Pi}$, we have

\begin{equation}
\chi_\pi(N_{k/k_0} g) = \chi_{\Pi}(\sigma \cdot g).
\end{equation}

Here $\chi_\pi$ and $\chi_{\Pi}$ are the characters of $\pi$ and $\tilde{\Pi}$. We will also call $\pi$ a (base change) descent or Shintani descent of $\Pi$ in this case.

Here characters are represented by functions which are locally integrable and locally constant on the set of regular semisimple elements [HC, Cl] and $N_{k/k_0} : G_k \rightarrow G_{k_0}$ is a norm map. If $G = GL$, $N_{k/k_0}$ is well defined up to conjugacy [AC]. However, for general $G$, since conjugacy classes are not stable with respect to field extensions [Ko], a norm map is not always well defined. Hence for the left hand side of (*) to be well defined, $\chi_\pi$ should be constant on stable conjugacy classes.

For the case $G = GL$, it is known that the Shintani lifting is surjective [AC, La] and it also coincides with Langlands functorial lift. However, as the examples of this paper show, in general, $\sigma$-invariant representations do not necessarily have Shintani descents to $G_{k_0}$. More precisely, we consider some representations of $GSp_4(k)$ of type $\theta_{10}$ (defined in §0.2) associated to a two dimensional algebra $K$ over $k$. These are analogous to $\theta_{10}$ of $Sp_4(k)$ [As, Sr]. Assuming that $K/k_0$ is a cyclic extension of fields (then $K/k_0$ is unramified or totally ramified), we prove that these representations of type $\theta_{10}$ are $\sigma$-invariant; however, they cannot be lifted from any admissible irreducible representation of $GSp_4(k_0)$ in the sense of Shintani base change. In the first part (I), we prove this by showing that $\chi_{\tilde{\theta}_{10}}$, the right hand side of (*), vanishes in a small neighborhood of $\sigma$ while the left hand side of (*) never vanishes in any small neighborhood of the identity.

In the second part (II), we compute the $L$-factor [PS] of $\theta_{10}$ associated to a quadratic unramified extension $K$ of $k$. In general, $L$-functions of supercuspidal