CHARACTERIZING $\omega_1$ AND THE LONG LINE
BY THEIR TOPOLOGICAL ELEMENTARY REFLECTIONS

BY

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ABSTRACT

Given a topological space $(X, T) \in M$, an elementary submodel of set theory, we define $X_M$ to be $X \cap M$ with the topology generated by $\{U \cap M : U \in T \cap M\}$. We prove that it is undecidable whether $X_M$ homeomorphic to $\omega_1$ implies $X = X_M$, yet it is true in ZFC that if $X_M$ is homeomorphic to the long line, then $X \setminus X_M$. The former result generalizes to other cardinals of uncountable cofinality while the latter generalizes to connected, locally compact, locally hereditarily Lindelöf $T_2$ spaces.

0. Introduction

We take $M$ to be an elementary submodel of $H_\theta$ for $\theta$ a sufficiently large regular cardinal, but act as if $H_\theta = V$. For an extended discussion of this standard circumlocution see [3] or [5] or [8].

Let $(X, T)$ be a topological space which is a member of $M$. Let $X_M$ be $X \cap M$ with topology $T_M$ generated by $\{U \cap M : U \in T \cap M\}$. In [8] the second author proved that if $X_M$ is homeomorphic to $\mathbb{R}$, then $X = X_M$. K. Kunen asked if analogous results hold for ordinals. The first section of this paper, which forms part of the University of Toronto Ph.D thesis of the first author [6], written under

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the supervision of the second author, shows that this is true for cardinals under an additional hypothesis, but is undecidable in general, even for $\omega_1$. This renders the second section result — due to the second author — quite surprising, namely that if $X_M$ is homeomorphic to the long line, i.e., $\omega_1 \times \mathbb{R}$ ordered lexicographically and given the order topology, then $X = X_M$.

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We need the following result:

**Theorem 0.1:** [2] Let $\langle X, \mathcal{T} \rangle$ be a locally compact $T_2$ space and let $M$ be a elementary submodel such that $\langle X, \mathcal{T} \rangle \in M$. Then there is a $Y \subseteq X$ and $\pi: \langle Y, \mathcal{T} \rangle \to X_M$ such that $\pi$ is perfect and onto.

The mapping is defined as follows: Let

$$V_x = \{ V \in \mathcal{T} \cap M : x \in V \}, \text{ for } x \in X_M.$$  

$$K_x = \bigcap V_x, \text{ for } x \in X_M.$$  

Note that, since $X$ is Hausdorff, a simple elementary submodel argument shows that if $x, y \in M$ and $x \neq y$, then $K_x \cap K_y = \emptyset$.

Define

$$Y = \bigcup \{ K_x : x \in X_M \},$$

and

$$\pi: \langle Y, \mathcal{T} \rangle \to \langle X_M, \mathcal{T}_M \rangle,$$

by

$$\pi(y) = x \text{ if and only if } y \in K_x.$$  

**1. Upwards reflection of cardinal spaces**

We first solve the easier question of what happens when $X_M$ is actually equal to an ordinal.

**Theorem 1.1:** Let $\kappa$ be an ordinal, $\langle X, \mathcal{T} \rangle$ a topological space and let $M$ be an elementary submodel such that $X, \mathcal{T}, \kappa \in M$. If $X_M = \kappa$ then $X = \kappa$.

**Proof:** Notice that as $X_M = \kappa$:

1. $M \models (\forall x, y \in X) (x \in y \text{ or } y \in x)$.
2. $M \models (\forall x \in X) (x \text{ is an ordinal})$.
3. $M \models (\forall x \in X) (\forall y \in x) (y \in X)$.  
