ON THE EXTENSION OF HÖLDER MAPS WITH VALUES IN SPACES OF CONTINUOUS FUNCTIONS

BY

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ABSTRACT

We study the isometric extension problem for Hölder maps from subsets of any Banach space into $c_0$ or into a space of continuous functions. For a Banach space $X$, we prove that any $\alpha$-Hölder map, with $0 < \alpha \leq 1$, from a subset of $X$ into $c_0$ can be isometrically extended to $X$ if and only if $X$ is finite dimensional. For a finite dimensional normed space $X$ and for a compact metric space $K$, we prove that the set of $\alpha$'s for which all $\alpha$-Hölder maps from a subset of $X$ into $C(K)$ can be extended isometrically is either $(0, 1]$ or $(0, 1)$ and we give examples of both occurrences. We also prove that for any metric space $X$, the above described set of $\alpha$'s does not depend on $K$, but only on finiteness of $K$.

1. Introduction–Notation

If $(X, d)$ and $(Y, \rho)$ are metric spaces, $\alpha \in (0, 1]$ and $K > 0$, we will say that a map $f: X \to Y$ is $\alpha$-Hölder with constant $K$ (or in short $(K, \alpha)$-Hölder) if

$$\forall x, y \in X, \quad \rho(f(x), f(y)) \leq K d(x, y)^\alpha.$$
Let us now recall and extend the notation introduced by Naor in [13]. For $C \geq 1$, $B_C(X, Y)$ will denote the set of all $\alpha \in (0, 1]$ such that any $(K, \alpha)$-Hölder function $f$ from a subset of $X$ into $Y$ can be extended to a $(CK, \alpha)$-Hölder function from $X$ into $Y$. If $C = 1$, such an extension is called an isometric extension. When $C > 1$, it is called an isomorphic extension. If a $(CK, \alpha)$-Hölder extension exists for all $C > 1$, we will say that $f$ can be almost isometrically extended. So, let us define:

$$A(X, Y) = B_1(X, Y), \quad B(X, Y) = \bigcup_{C \geq 1} B_C(X, Y) \quad \text{and} \quad \bar{A}(X, Y) = \bigcap_{C > 1} B_C(X, Y).$$

The study of these sets goes back to a classical result of Kirszbraun [10] asserting that if $H$ is a Hilbert space, then $1 \in A(H, H)$. This was extended by Grünbaum and Zarantonello [5] who showed that $A(H, H) = (0, 1]$. Then the complete description of $A(L^p, L^q)$ for $1 < p, q < \infty$ relies on works by Minty [12] and Hayden, Wells and Williams [6] (see also the book of Wells and Williams [14] for a very nice exposition of the subject). More recently, K. Ball [1] introduced a very important notion of non-linear type or cotype and used it to prove a general extension theorem for Lipschitz maps. Building on this work, Naor [13] improved the description of the sets $B(L^p, L^q)$ for $1 < p, q < \infty$.

In this paper, we concentrate on the study of $A(X, Y)$ and $\bar{A}(X, Y)$, when $X$ is a Banach space and $Y$ is a space of converging sequences or, more generally, a space of continuous functions on a compact metric space. This can be viewed as an attempt to obtain a non-linear version of the results of Lindenstrauss and Pelczyński [11] and later of Johnson and Zippin ([8] and [9]) on the extension of linear operators with values in $C(K)$ spaces.

So let us denote by $c$ the space of all real converging sequences equipped with the supremum norm and by $c_0$ the subspace of $c$ consisting of all sequences converging to 0. If $K$ is a compact space, $C(K)$ denotes the space of all real valued continuous functions on $K$, equipped again with the supremum norm.

In section 2, we show that if $X$ is infinite dimensional and $Y$ is any separable Banach space containing an isomorphic copy of $c_0$, then $\bar{A}(X, Y)$ is empty. On the other hand, we prove that $A(X, c_0) = (0, 1]$, whenever $X$ is finite dimensional.

In section 3, we show that for any finite dimensional space $X$, $\bar{A}(X, c) = (0, 1]$ and $A(X, c)$ contains $(0, 1)$. Then the study of the isometric extension for Lipschitz maps turns out to be a bit more surprising. Indeed, we give an example of a 4-dimensional space $X$ such that $A(X, c) = (0, 1)$. To our knowledge, this provides the first example of Banach spaces $X$ and $Y$ such that $A(X, Y)$ is not