1 Introduction

In this paper, we study properties of positive solutions of semilinear elliptic equations with critical exponent. We give different proofs, improvements, and extensions to some previously established Liouville-type theorems and Harnack-type inequalities.

For \( \mu > 0, \bar{x} \in \mathbb{R}^n, \ n \geq 3, \)

\[
(1) \quad u(x) = \left( \frac{\mu}{1 + \mu^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}}
\]
satisfies

\[
(2) \quad -\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \quad u > 0, \text{ in } \mathbb{R}^n.
\]

The following celebrated Liouville-type theorem was established by Caffarelli, Gidas, and Spruck.

**Theorem 1.1** (\cite{12}). A \( C^2 \) solution of (2) is of the form (1).

Under the additional hypothesis \( u(x) = O(|x|^{2-n}) \) for large \( |x| \), the result was established earlier by Obata \([49]\) and Gidas, Ni and Nirenberg ([30]). The proof of Obata is more geometric, while the proof of Gidas, Ni and Nirenberg is by the method of moving planes. The proof of Caffarelli, Gidas and Spruck is by a "measure theoretic" variation of the method of moving planes. Such Liouville-type
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Theorems have played a fundamental role in the study of semilinear elliptic equations with critical exponent, which include the Yamabe problem and the Nirenberg problem. The method of moving planes (and its variants including the method of moving spheres, etc.) goes back to A. D. Alexandroff in his study of embedded constant mean curvature surfaces. It was then used and developed through the work of Serrin ([54]) and Gidas, Ni and Nirenberg ([30] and [31]). In recent years, and stimulated by a series of beautiful papers of Berestycki, Caffarelli and Nirenberg ([11]–[8]), the method has been widely used and has become a powerful and user-friendly tool in the study of nonlinear partial differential equations. In this paper, we develop a rather systematic, and simpler, approach to Liouville-type theorems and Harnack-type inequalities along the lines of [42] and [26] using the method of moving spheres.

For \( n > 3 \), let \( \mathbb{R}^n_+ = \{ x = (x', t) ; x' \in \mathbb{R}^{n-1}, t > 0 \} \) denote Euclidean half space. For \( \mu > 0 \), \( \bar{x} = (\bar{x}', \bar{t}) \in \mathbb{R}^n, \)

\[
(3) \quad u(x', t) = \left( \frac{\mu}{1 + \mu^2 |(x', t) - (\bar{x}', \bar{t})|^2} \right)^{\frac{n-2}{2}}
\]
satisfies

\[
(4) \begin{cases}
-\Delta u = n(n - 2)u^{\frac{n+2}{n-2}}, & u > 0, \quad \text{in } \mathbb{R}^n_+,
\frac{\partial u}{\partial t} = cu^{\frac{n}{n-2}}, & \text{on } \partial \mathbb{R}^n_+,
\end{cases}
\]

where \( c = (n - 2)\mu \).

The following theorem was established by Li and Zhu.

**Theorem 1.2 ([42]).** A \( C^2 \) solution of (4) is of the form (3) for some \( \mu > 0 \), \( \bar{x}' \in \mathbb{R}^{n-1} \), and \( \bar{t} = \frac{c}{(n-2)\mu} \).

Under an additional hypothesis \( u(x) = O(|x|^{2-n}) \) for large \( |x| \), the result was established earlier by Escobar ([28]). The proof of Escobar is along the lines of the proof of Obata, while the proof of Li and Zhu is by the method of moving spheres, a variant of the method of moving planes.

Liouville-type theorems in dimension \( n = 2 \) were established in [22], [27], [42], and the references therein. Analogues for systems were established in [14]. Improvements to the results in [42] can be found in recent papers of Ou ([51]) and the second author ([55]).

For \( n \geq 3 \), Liouville-type theorems for more general semilinear equations

\[
(5) \quad -\Delta u = g(u), \quad u > 0, \quad \text{in } \mathbb{R}^n,
\]