ON SOLUTIONS OF THE BELTRAMI EQUATION

By

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Abstract. In this paper we study the existence and uniqueness of solutions of the Beltrami equation \( f_\bar{z} = \mu(z) f_z \), where \( \mu(z) \) is a measurable function defined almost everywhere in a plane domain \( \Delta \) with \( \|\mu\|_{\infty} = 1 \). Here the partials \( f_\bar{z} \) and \( f_z \) of a complex valued function \( f(z) \) exist almost everywhere. In case \( \|\mu\|_{\infty} \leq q < 1 \), it is well-known that homeomorphic solutions of the Beltrami equation are quasiconformal mappings. In case \( \|\mu\|_{\infty} = 1 \), much less is known. We give sufficient conditions on \( \mu(z) \) which imply the existence of a homeomorphic solution of the Beltrami equation, which is ACL and whose partial derivatives \( f_\bar{z} \) and \( f_z \) are locally in \( L^q \) for any \( q < 2 \). We also give uniqueness results. The conditions we consider improve already known results.

1 Introduction

In the Beltrami equation

\[
(\beta) \quad f_\bar{z}(z) = \mu(z) f_z(z),
\]

\( \mu(z) \) is to be a measurable function defined almost everywhere in a plane domain \( \Delta \) with \( \text{ess.l.u.b.} \|\mu\|_{\infty} = 1 \). Here the partials \( f_\bar{z} \) and \( f_z \) of a complex valued function \( f(z) \) exist almost everywhere. In case \( \|\mu\|_{\infty} \leq q < 1 \), it is well-known that homeomorphic solutions of the Beltrami equation are quasiconformal mappings with maximum dilatation

\[
D(z) \leq K = \frac{1 + q}{1 - q}.
\]

In case \( \|\mu\|_{\infty} = 1 \), much less is known. The only significant results known to the authors are due to O. Lehto [4] and [5] and G. David [2].

In [4] Lehto treats the case of the plane with the following two stringent restrictions on \( \mu(z) \):

(A1) in the complement of a compact set of measure 0, \( |\mu| \) is bounded away from 1 on every compact subset;

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(Λ₂) for any complex z and 0 < r₁ < r₂ < ∞,

\[ \int_{r₁}^{r₂} \left( 1 + 2\pi \int_0^{2\pi} \frac{1 - e^{-2i\theta} \mu(z + re^{i\theta})^2}{1 - |\mu(z + re^{i\theta})|^2} d\theta \right)^{-1} \frac{dr}{r} \]

is strictly positive and tends to ∞ as \( r₁ \to 0 \) or \( r₂ \to \infty \).

Under these conditions, he proves the existence of a homeomorphic solution to the Beltrami equation. Condition (Λ₂) essentially supposes the pointwise equicontinuity of the approximating functions to be discussed below, thus bypassing the most difficult step in the existence proof. In [4] and [5] Lehto does not study the question of uniqueness.

In [2] David follows through the proof of existence in Ahlfors’ monograph [1], giving some very detailed and complicated estimates. He considers the case when \( \mu \) is defined in the plane and assumes that

(Δ) there exist constants \( a > 0 \) and \( C > 0 \) such that for \( \epsilon > 0 \) sufficiently small

\[ \text{measure}\{z : |\mu(z)| > 1 - \epsilon\} \leq Ce^{-\alpha/\epsilon}. \]

Under this condition he proves the existence of a homeomorphic solution and shows that under suitable normalizations it satisfies a uniqueness result.

The main results as well as various auxiliary results in this paper are obtained under conditions of the form

\[ \iint_B F \left( \frac{1}{1 - |\mu(z)|} \right) dA < \Phi_B, \]

where \( B \) is a bounded measurable set, \( \Phi_B > 0 \) is a constant which depends on \( B \), and \( F(x) \), defined for \( x \geq 1 \), is either the identity function, or \( F(x) = x^\lambda, \lambda > 1, \) or

\[ F(x) = \exp \frac{x}{1 + \log x}. \]

With the choice

\[ F(x) = \exp \frac{x}{1 + \log x}, \]

i.e., condition (A) below, we prove the existence of a homeomorphic solution of the Beltrami equation having properties detailed in the statement of our Theorem 1 (Existence Theorem). We also give uniqueness results, which are stated as Theorems 2 and 2’.

In the Appendices we compare our results with those already known. We show that David’s results are not subsumed by Lehto’s by providing an example where (Δ) holds while (Λ₁) does not. We also show that (Δ) implies conditions (A) and (B) of Theorem 1 but that condition (A) does not imply condition (Δ).