WHICH PERTURBATIONS OF QUASIANALYTIC WEIGHTS PRESERVE QUASIANALYTICITY?
HOW TO USE DE BRANGES' THEOREM

By

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Abstract. Based on an approach of de Branges and the theory of entire functions, we prove two results pertaining to the Bernstein approximation problem, one concerning analytic perturbations of quasianalytic weights and the other dealing with density of polynomials in spaces with nonsymmetric weights. We improve earlier results of V. P. Gurarii and A. Volberg, giving a more complete answer to a question posed by L. Ehrenpreis and S. N. Mergelyan.

A nonnegative function \( \varphi(x) \) defined on the real line is called a weight if it is continuous on its support, \( \varphi \in C(E_\varphi), E_\varphi = \text{supp}(\varphi) \), and
\[
|\varphi(x)| \to 0 \quad \text{as} \quad |x| \to \infty, \quad n = 0, 1, 2, \ldots.
\]
In the usual way we define the space \( C^0_\varphi \) which consists of functions \( f \in C(E_\varphi) \) such that
\[
\varphi(x)|f(x)| \to 0 \quad \text{as} \quad |x| \to \infty.
\]
Then \( C^0_\varphi \) contains the polynomials and is a Banach space in the norm
\[
\|f\|_\varphi = \sup_{x \in E_\varphi} \varphi(x)|f(x)|.
\]

We say that the weight \( \varphi(x) \) is quasianalytic if the polynomials are dense in \( C^0_\varphi \) or, what is the same (by the Hahn–Banach theorem), if the class of Fourier transforms
\[
\hat{\mu}_\varphi(t) = \int_{E_\varphi} e^{int}\varphi(x) \, d\mu(x), \quad \|\mu\| < \infty
\]
is quasianalytic in the sense of Hadamard, i.e. if, for some \( t_0 \in \mathbb{R} \), \( \hat{\mu}_\varphi^{(n)}(t_0) = 0, \quad n = 0, 1, 2, \ldots \), then \( \hat{\mu}_\varphi(t) \equiv 0 \) on \( \mathbb{R} \) and hence \( \mu \equiv 0 \).

We say that a quasianalytic weight is normal if it remains quasianalytic after its redefinition on an arbitrary set consisting of a finite number of points. Otherwise, we say that a quasianalytic weight is singular. An equivalent definition says that the
quasianalytic weight $\varphi$ is normal if, for each $n = 1, 2, \ldots$, the weight $(1 + |x|)^n \varphi(x)$ is also quasianalytic (see [18, §5]).

S. N. Mergelyan proved in [18, §5] that each singular quasianalytic weight is always supported by the zero set of an entire function of zero exponential type (this follows also from the proof of Theorem 1 below). We note that this condition is not sufficient. For example, the weight

$$\varphi(x) = \begin{cases} \frac{1}{|B_\alpha'(x)|}, & x = 1, 2^\alpha, 3^\alpha, \ldots \\ 0, & x \neq 1, 2^\alpha, 3^\alpha, \ldots \end{cases}$$

where

$$B_\alpha(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^\alpha}\right), \quad \alpha > 2,$$

studied by A. E. Fryntov in [8], is quasianalytic and normal.

Here, we study when certain perturbations of a normally quasianalytic weight preserve quasianalyticity. In particular, we investigate the case in which a normal quasianalytic weight $\varphi$ is supported on a half-line, say $\text{supp}(\varphi) \subset [0, \infty)$, while the perturbation $\varphi_0$ is supported on the complementary half-line, $\text{supp}(\varphi_0) \subset (-\infty, 0]$.

1. Main results

Theorem 1 Let $\varphi(x)$ be a normal quasianalytic weight and let $\varphi_0(x)$ be a weight such that for some $\delta > 0$

$$\lim_{|x| \to \infty} e^{\delta |x|} \varphi_0(x) = 0.$$

Then the weight $\varphi + \varphi_0$ is normally quasianalytic as well.

Condition (2) means that the Fourier transform (1) with $\varphi_0$ instead of $\varphi$ is analytic in the strip $\{\text{Im } z < \delta\}$. Thus Theorem 1 says that each analytic perturbation of a normal quasianalytic weight is still quasianalytic (and in fact normal). It should be mentioned here that N. Levinson studied related problems for classes of Fourier transforms which cannot vanish on an interval [14, Theorems XXIII, XXV, XXVII].

In spite of a great deal of effort (see surveys by N. I. Akhiezer [1], S. N. Mergelyan [18], B. Ya. Levin [13], and P. Koosis’ book [11]) not so many cases are known where density of polynomials can be verified explicitly in terms of the weight $\varphi$. If $\varphi(x) = 1/W(x)$, $x \in \mathbb{R}$, where $W(x)$ is an even increasing function and $\log W(e^t)$ is convex, then the polynomials are dense in $C^0_{w-1}$ iff

$$\int_{-\infty}^{+\infty} \frac{\log W(x)}{1 + x^2} \, dx = +\infty.$$