ON THE EXISTENCE OF MATRICES WITH PRESCRIBED HEIGHT AND LEVEL CHARACTERISTICS

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ABSTRACT

We determine all possible relations between the height (Weyr) characteristic and the level characteristic of an $M$-matrix. Under the assumption that the two characteristics have the same number of elements, we determine the possible relations between the two characteristics for a wider class of matrices, which also contains the class of strictly triangular matrices over an arbitrary field. Given two sequences which satisfy the above condition, we construct a loopless acyclic graph $G$ with the following property: Every matrix whose graph is $G$ has its height characteristic equal to the first sequence and its level characteristic equal to the second. We give several counterexamples to possible extensions of our results, and we raise some open problems.

1. Introduction

In this paper we determine all possible relations between the height (Weyr) characteristic and the level characteristic of an $M$-matrix. Indeed, under the assumption that the two characteristics have the same number of elements, we

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determine the possible relations between the two characteristics for a wider class of matrices, which also contains the class of strictly triangular matrices over an arbitrary field.

In the introduction to [15] it is observed that there is a relation between the two characteristics for an $M$-matrix, and in that paper and in several subsequent ones the case of equality was explored, e.g [13], [12], [1], [8], [9]. The question of characterizing all possible relations was thus raised implicitly in the 1950's, and the problem was stated explicitly in [16]. In this paper we solve this problem. Given two sequences $\eta$ and $\lambda$ of positive integers with the same number of elements, we show that there exists a matrix in the class considered with level characteristic $\lambda$ and height characteristic $\eta$ if and only if $\eta$ majorises $\lambda$ when reordered in non-increasing order. This characterization was conjectured by Berman and van den Driessche [private communication, 1987].

We now describe our paper in more detail. Section 2 is devoted to definitions and notation. In particular we here define the level characteristic and the height characteristic of $A$. We also give our definition of majorization which is related to the definition found in many places, e.g. [11, p.7], but is not identical with it, since we wish to be consistent with our definition in [8].

In Section 3 we study the relation between the height and level characteristics for strictly lower triangular matrices under the assumption that the characteristics have the same number of elements. Given two sequences $\eta$ and $\lambda$ which satisfy the above condition, we construct a loopless acyclic graph $G$ with the following property: Every matrix whose graph is $G$ has its height characteristic equal to $\eta$ and its level characteristic equal to $\lambda$. This result is then extended in Section 4 to the class of matrices all of whose singular vertices are simple (i.e. matrices which may be partitioned into a block triangular form, so that the singular diagonal blocks have 0 as a simple eigenvalue). This class, of course, contains the class of $M$-matrices. We thus obtain the results mentioned above. The paper is concluded with some partial results on the two characteristics for the class of matrices with all singular vertices simple, where we omit the assumption that the characteristics have the same number of elements.

In Sections 3 and 4 we give several examples which illustrate our theorems and we state counterexamples to possible extensions of our results. We also raise some open problems. For example, given sequences $\lambda$ and $\eta$, it would be interesting to characterize all graphs $G$ such that for all matrices $A$ whose graph is $G$, the level characteristic of $A$ equals $\lambda$ and the height characteristic equals $\eta$.

This paper continues the series of joint papers [6], [7], [4], [5], [8], and [9],