The Dimension of Space-Time.

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Summary. – The standard assumptions about the basic principles of quantum mechanics, rotational (and Lorentz) invariance, the real nature of space-time coordinates, and the complex nature of state functions of particles, are related to the fact that space-time has $3 + 1$ dimensions.

One of the basic properties of physical space-time is its dimensionality $3 + 1$. Attempts have been made to explain the dimension (1,2), and the reasons presented are interesting, and may have significant implications. Here we should like to consider another aspect of the question, how the dimensionality might be related to quantum mechanics.

Of course, the significance of any relationship depends on what in quantum mechanics we consider important, for dropping or adding assumptions makes the connection more or less interesting. Here we consider only the standard assumptions and views, to see to what extent the dimension can be related to these. These assumptions are used because they seem to be the ones which are the most generally accepted, at least as being the simplest. They are not analyzed here, and clearly such analysis is important. However, our present purpose is only to see how the most ordinary assumptions are related to the question of the dimensionality, not to consider the extent to which these assumptions need hold.

The approach is based on the following points.

I) Space-time is real; its co-ordinates are real numbers.

II) The objects in it are described by functions over space-time, and these functions are complex. This is one point at which quantum mechanics enters.


III) The properties of space-time (that is the laws of physics) are the same for any two relatively rotated observers. (This statement is probably stronger than necessary.)

Note that only the very basic aspects of quantum mechanics, not its detailed structure is used. Also rotations includes space-time rotations.

These points are certainly well-grounded in experiment and our current understanding of physics. How can these be related to the dimension?

Space-time is real so the transformations in it leaving a point fixed form an orthogonal group. State functions are complex, so the corresponding transformations form a unitary group.

Suppose now that we have a set of identical observers observing a particle. Some of these are rotated. The particle looks somewhat different to them; its state function is different. But the new state function and the old are related and the relationship depends on the particular rotation.

The rotations of the observers, and the transformations of the state functions, can each be generated from a set of generators. In order for the laws of physics to look the same to all the variously rotated observers we would expect that for each generator of a rotation there corresponds a generator of a state function transformation. Further we would expect that the commutator of the two transformations corresponds to the commutator of the two rotations to which each of the two transformations corresponds.

The rotations of the observers, being rotations of the co-ordinate system, form an orthogonal group. The transformations of the complex state functions form a unitary group. So we would expect that the unitary and orthogonal algebras are isomorphic (which would seem to be the standard assumption, but which clearly requires a detailed analysis).

Thus we would expect space-time to be such that the algebras of the orthogonal groups over it are isomorphic to algebras of unitary groups.

What orthogonal algebras are isomorphic to unitary algebras? The answer is given implicitly by RACAH (3).

In general, for a unitary algebra \((A(k))\) to be isomorphic to an orthogonal algebra \((B(k), D(k))\), the ranks (the \(k\)'s) must be equal, and the orders of the algebras must be equal. The orders are given by RACAH, and are \(A: (k+1)^2 - 1\), \(B: k(2k+1)\), \(D: k(2k-1)\). Thus \((k+1)^2 - 1 = k(2k \pm 1)\). This gives \(k = 1\) (for \(B\)), and \(3\) (for \(D\)), and \(k = 0\).

For \(k = 0\) there is no algebra on the list, but the algebras of \(U_1\) and \(O_0\) are isomorphic, both having a single generator. A two-dimensional real space admits a one-parameter group which is homomorphic to the unitary group over a one-dimensional complex space, the one-parameter group of phase changes.

For \(k = 1\) there is the isomorphism between the algebras of \(SU_2\) and \(O_3\), giving a corresponding group homomorphism. Thus 3-dimensional space is a possibility. The algebras of \(SU_{1,1}\) and \(O_{2,1}\) are also isomorphic, so a \((2+1)\)-dimensional space is another possibility.

For \(k = 3\) there is an isomorphism between the algebras of \(SU_4\) and \(O_4\). However, a six-dimensional space has five-dimensional subspaces, and for these there is no corresponding isomorphism. Thus we do not consider here a six-dimensional space.

Besides these algebras over real numbers the algebras over complex numbers must be considered. The algebra of \(O_4\) is not simple, being the direct sum of the algebras of \(O_3 \times O_3\). The space decomposes into two completely independent subspaces, so it

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