SUCCESSOR OF SINGULARS:
COMBINATORICS AND NOT COLLAPSING
CARDINALS ≤ κ IN (< κ)-SUPPORT ITERATIONS

BY

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ABSTRACT
On the one hand, we deal with (< κ)-supported iterated forcing notions which are (E0, C1)-complete, bearing in mind problems on Whitehead groups, uniformizations and the general problem. We deal mainly with the case of a successor of the singular cardinal. This continues [Sh 587].

On the other hand, we deal with complimentary ZFC combinatorial results.

Annotated Contents

§1. GCH implies for successor of singular no stationary S has uniformization ............................ 128

[For λ strong limit singular, for stationary S ⊆ S^λ^+_{cf(λ)} we prove strong negation of uniformization for some S-ladder system and even weak versions of diamond. E.g., if λ is singular strong limit and 2^λ = λ^+, then there are γ_i^δ < δ increasing in i < cf(λ) with limit δ for each δ ∈ S such that for every f: λ^+ → α^* < λ for stationarily many δ ∈ S, for every i we have f(κ_2i) = f(κ_2i+1).]

§2. Case C: Forcing for successor of singulars ......................... 131

[Let λ be strong limit singular κ = λ^+ = 2^λ, S ⊆ S^κ_{cf(λ)} stationary not reflecting. We present the consistency of a forcing axiom implying, e.g.: if h_δ is a function from A_δ to θ, A_δ ⊆ δ = sup(A_δ),

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§3. $\kappa^+$.c.c. and $\kappa^+$.pic ........................................... 141

[In the forcing axioms we would like to allow forcing notions of cardinality $>\kappa$; for this we use a suitable chain condition (allowed here and in [Sh 587]). This sheds more light on the strongly inaccessible case and we comment on this (and forcing against cases of diamonds).]

§4. Existence of non-free Whitehead (and Ext($G, \mathbb{Z}) = 0$) abelian groups in successor of singulars ........................................ 150

[We use the information on the existence of weak version of the diamond for $S \subseteq S^\lambda_\lambda^+$, $\lambda$ strong limit singular with $2^\lambda = \lambda^+$, to prove that there are some abelian groups with special properties (from reasonable assumptions). We also get more combinatorial principles on $\lambda = \mu^+$, $\mu > \text{cf}(\mu)$ (even if just $\lambda = 2^{2^\sigma}$).]

References .................................................. 155

§1. GCH implies for successor of singular no stationary $S$ has uniformization

We show that a major improvement in [Sh 587] over [Sh 186] for inaccessible (every ladder on $S$ has uniformization rather than some ladder on $S$) cannot be done for successor of singulars. This is continued in §4.

1.1 FACT: Assume

(a) $\lambda$ is strong limit singular with $2^\lambda = \lambda^+$, let $\text{cf}(\lambda) = \sigma$

(b) $S \subseteq \{\delta < \lambda^+: \text{cf}(\delta) = \sigma\}$ is stationary.

Then we can find $(< \gamma^\delta_i : i < \sigma : \delta \in S)$ such that

(\alpha) $\gamma^\delta_i$ is increasing (with $i$) with limit $\delta$

(\beta) if $\mu < \lambda$ and $f : \lambda^+ \to \mu$ then the following set is stationary:

$\{\delta \in S : f(\gamma^\delta_{2i}) = f(\gamma^\delta_{2i+1})$ for every $i < \sigma\}$.

Moreover

(\beta') if $f_i : \lambda^+ \to \mu_i, \mu_i < \lambda$ for $i < \sigma$ then the following set is stationary:

$\{\delta \in S : f_i(\gamma^\delta_{2i}) = f_i(\gamma^\delta_{2i+1})$ for every $i < \sigma\}$.

Proof: This will prove 1.2, too. We first concentrate on (\alpha) + (\beta) only.

Let $\lambda = \sum_{i < \sigma} \lambda_i, \lambda_i$ a cardinal increasing continuous with $i, \lambda_{i+1} > 2^\lambda, \lambda_0 > 2^\sigma$. For $\alpha < \lambda^+$, let $\alpha = \bigcup_{i < \sigma} a_{\alpha,i}$ such that $|a_{\alpha,i}| < \lambda_i$. Without loss of generality $\delta \in S \Rightarrow \delta$ divisible by $\lambda^\omega$ (ordinal exponentiation). For $\delta \in S$ let $< \beta^\delta_i : i < \sigma$ be increasing continuous with limit $\delta$, $\beta^\delta_i$ divisible by $\lambda$ and $> 0$.

For $\delta \in S$ let $< \beta^\delta_i : i < \sigma$ be such that: $b^\delta_i < \beta^\delta_i, |b^\delta_i| < \lambda_i, b^\delta_i$ is increasing