MAXIMAL ERGODIC THEOREMS AND APPLICATIONS TO RIEMANNIAN GEOMETRY

BY

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The first author dedicates this paper to his parents José Martiniano and Zoraide

ABSTRACT

We prove new ergodic theorems in the context of infinite ergodic theory, and give some applications to Riemannian and Kähler manifolds without conjugate points. One of the consequences of these ideas is that a complete manifold without conjugate points has nonpositive integral of the infimum of Ricci curvatures, whenever this integral makes sense. We also show that a complete Kähler manifold with nonnegative holomorphic curvature is flat if it has no conjugate points.

0. Introduction

Infinite ergodic theory is the study of measure-preserving transformations of infinite measure spaces. A class of very natural examples is that of null-recurrent Markov chains (resp. their shifts) such as the symmetric coin-tossing random walk on the integers. There is a great variety of ergodic behavior infinite measure-preserving transformations can exhibit, and they have undergone some intense research within the last twenty years, much of which is associated with the name of Aaronson ([Aa]). In his book Aaronson studied the standard σ-finite measure spaces and non-singular measure preserving transformations.

This paper will provide another class of natural examples in the category of infinite ergodic theory. We prove new maximal ergodic theorems, which include

* Both authors are partially supported by CNPq, Brazil.
Received January 28, 2002
some known geometric results and have some new geometric consequences. We will restrict the discussion on manifolds even though some of the results can be generalized to more general cases.

Let $N$ be a manifold equipped with the $\sigma$-algebra $\beta$ of Borel sets, a flow $T_t$, $t \in \mathbb{R}$, and a $T_t$-invariant measure $\mu$. Let $g: N \to \mathbb{R}$ be a measurable function. We say that a measurable function $g$ has **well-defined integral** if either the positive or the negative part of $g$ is integrable on $M$. We start with the statement of the classical Maximal Ergodic Theorem (see [Pt], here we use “inf” instead of the usual “sup” for later convenience in applications).

**MAXIMAL ERGODIC THEOREM:** Let $g$ be a measurable function, with well-defined integral on $N$, and $Z \subset N$ be a $T_t$-invariant Borel subset. Set

$$E[g] = \left\{ w \in Z \mid \inf_{s>0} \int_0^s g(T_t w) dt \leq 0 \right\}.$$

Then $\int_{E[g]} g d\mu \leq 0$.

Our main ergodic result is the following new maximal ergodic theorem.

**THEOREM 0.1:** Let $f$ be a measurable function with well-defined integral on $N$ and $Z \subset N$ be a $T_t$-invariant Borel set. Consider the following $T_t$-invariant subset,

$$E(f) = \left\{ w \in Z \mid \liminf_{s \to \pm \infty} \int_{I_s} f(T_t w) dt \leq 0, \liminf_{s \to \pm \infty} \int_{I_s} f(T_t w) dt < +\infty \right\},$$

where $I_s$ denotes the interval $[0, s]$ if $0 < s$, and $[s, 0]$ if $s < 0$. Then $\int_{E(f)} f d\mu \leq 0$.

Here we allow the time to go to infinity in both directions and the total measure spaces to be infinite. Thus many results in ergodic theory can be reformulated under this point of view. As shown by our applications to Riemannian geometry, it is particularly useful in spaces of infinite measure.

When the measure of $N$ is finite, it is not difficult to obtain Theorem 0.1 from the Maximal Ergodic Theorem. So the importance of Theorem 0.1 relies on the ergodic theory on infinite measure spaces. We will also give in the first section another version of Theorem 0.1 (see Theorem 1.2) with the conservative and dissipative parts of $N$ separated.

As a corollary of Theorem 0.1 we can obtain a pointwise ergodic Theorem.

**COROLLARY 0.1:** Let $f$ be a measurable function with well-defined integral on some $T_t$-invariant Borel subset $E$. Assume that for almost all $w \in E$ we have

$$\liminf_{s \to \pm \infty} \int_{I_s} f(T_t w) dt < +\infty, \limsup_{s \to \pm \infty} \int_{I_s} f(T_t w) dt > -\infty.$$