AN APPLICATION OF THE ALMOST-POSITIVITY
OF A CLASS OF FOURTH-ORDER
PSEUDODIFFERENTIAL OPERATORS

By
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Abstract. Using the almost-positivity of a class of fourth-order pseudodifferential operators, we prove the inequality

$$||Lu||_{(0)} + ||u||_{(0)} \geq C_K \left( ||u||_{(1/2)} + ||u||_{(1)} \right), \quad C_K > 0, \quad \forall u \in C_0^\infty(K),$$

for any compact set $K \subset \Omega$, an open set of $\mathbb{R}^n$, where $L = L^* \in \Psi^2_{phg}(\Omega)$ has principal symbol $p_2 \geq 0$ transversally elliptic with respect to the characteristic manifold $\Sigma = p_2^{-1}(0)$, the condition

$$p_1^T(\rho) + \text{Tr} F p_2(\rho) > 0$$

is satisfied on $\Sigma$, and where $X \in \Psi^4_{phg}(\Omega)$ has principal symbol vanishing on $\Sigma$.

Applications to the case $L = \sum_{j=1}^n X_j^2 X_j + X_0$, where $X_0, X_j \in \Psi^4_{phg}(\Omega)$, with $X_j, X$ complex-valued, are given.

1. Introduction

In the paper [8] a very special class of properly supported pseudodifferential operators with characteristics of multiplicity 4 was considered and, for $P = P^* \in \Psi^4_{phg}(\Omega)$ (\Omega an open subset of $\mathbb{R}^n$) belonging to such a class, necessary and sufficient conditions were given in order for

$$(Pu, u) \geq -C_K ||u||_{(1/2)}^2, \quad \forall u \in C_0^\infty(K)$$

to hold, where $K$ is any compact subset of $\Omega$, and $C_K > 0$.

These conditions were given in terms of conditions on $p_3^s(x, \xi)$, the subprincipal symbol of $P$ defined by $p_3^s(x, \xi) = p_3(x, \xi) + i(\partial_x, \partial_\xi)p_4(x, \xi)/2$, and on the second-order part of the Weyl-symbol of $P$, $\sigma^s_2(P)(x, \xi) = p_2(x, \xi) + i(\partial_x, \partial_\xi)p_3(x, \xi)/2 - (\partial_x, \partial_\xi)^2 p_4(x, \xi)/8$, at points of $\Sigma$, the characteristic set of the principal symbol of $P$, where the symbol of $P$ is $p(x, \xi) \sim p_4(x, \xi) + p_3(x, \xi) + p_2(x, \xi) + \cdots$.

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In the sequel, given a real quadratic form \( Q(v, v), v \) belonging to a real symplectic vector space \((V, \sigma)\), we shall associate with it, as usual, the \textit{fundamental matrix} by means of the identity

\[
Q(v, w) = \sigma(v, Fw), \quad v, w \in V.
\]

\( \text{Tr}^+ F \) will then denote the \textit{positive trace} of \( F \), namely \( \sum_{\mu_j > 0} \mu_j \), with \( i\mu_j \) in the spectrum of \( F \).

We now briefly recall the main hypotheses of [8]. Let \( P = Q^2 + R \), where \( Q = Q^* \in \Psi^2_{phg}(\Omega), R = R^* \in \Psi^3_{phg}(\Omega) \) are \textit{properly supported} and suppose

(i) \( q_2 \geq 0 \) vanishes on \( \Sigma = q_2^{-1}(0) \), supposed a \( C^\infty \)-manifold on which the symplectic form \( \sigma \) of \( T^*\Omega \) has \textit{constant rank}, and the \textit{radical} of the Hessian of \( q_2 \) at \( \rho \) is \( T_\rho \Sigma \), for any \( \rho \in \Sigma \) (this condition is classically referred to as \textit{transversal ellipticity} of \( q_2 \) with respect \( \Sigma \); see [6], page 361).

(ii) \( r_3(\rho) = 0, \quad dr_3(\rho) = 0 \) when \( \rho \in \Sigma \).

The hypotheses imply that the Hessians of \( q_2/2 \) and \( r_3/2 \) are invariantly defined on \( \Sigma \), and that \( \text{Ker}F_Q(\rho) = T_\rho \Sigma \), where we denote by \( F_Q(\rho) \) and \( F_R(\rho) \) the fundamental matrices associated with the Hessians of \( q_2/2 \) and \( r_3/2 \), respectively, at the point \( \rho \in \Sigma \). It was then required that

(iii) \[
[F_Q(\rho), F_R(\rho)] = 0, \quad \forall \rho \in \Sigma.
\]

Notice that in this case \( \text{Im}F_Q(\rho)^2 \) and \( \text{Ker}F_Q(\rho)^2 \) are symplectic, \textit{invariant} subspaces for \( F_R(\rho) \). (We shall not mention here the last hypothesis that is made in [8] (namely hypothesis \( (H5) \) of the quoted paper) since in our applications the set \( \Sigma_c \) introduced in [8] will always be empty.)

Note that in this case one has (with \( q^1_1 \) denoting the subprincipal symbol of \( Q \)), for any \( \rho \in \Sigma \),

\[
F_{q^1_1}(\rho) = 2q^1_1(\rho)F_Q(\rho) + F_R(\rho) \quad \text{and} \quad [F_{q^1_1}(\rho), F_Q(\rho)] = 0.
\]

In order to state the result of [8], we give the following definition.

**Definition** 0 Let the fundamental matrices \( F_Q \) and \( F_{q^1_1} \) be commuting on \( \Sigma \). We say that the quadratic form \( \sigma(v, F_{q^1_1}(\rho)v) \) is \textit{not too negative compared to} \( \sigma(v, F_Q(\rho)v), v \in T_\rho(T^*\Omega), \rho \in \Sigma \), if and only if one of the following occurs:

(a) \[
s(\rho) \geq -2\text{Tr}^+ F_Q(\rho),
\]

(b) \[
s(\rho) = \min_{v \in (\text{Ker}F_Q(\rho)^2)/\text{Ker}F_Q(\rho)} \frac{\sigma(v, F_{q^1_1}(\rho)v)}{\sigma(v, F_Q(\rho)v)},
\]