AN APPLICATION OF THE ALMOST-POSITIVITY
OF A CLASS OF FOURTH-ORDER
PSEUDODIFFERENTIAL OPERATORS

By
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Abstract. Using the almost-positivity of a class of fourth-order pseudodifferential operators, we prove the inequality

\[ \|Lu\|_{(0)} + |u|_{(0)} \geq C_K \left( \|Xu\|_{(1/2)} + |u|_{(1)} \right), \quad C_K > 0, \quad \forall u \in C^\infty_0(K), \]

for any compact set \( K \subset \Omega \), an open set of \( \mathbb{R}^n \), where \( L = L^* \in \Psi^2_{phg}(\Omega) \) has principal symbol \( p_2 \geq 0 \) transversally elliptic with respect to the characteristic manifold \( \Sigma = p_2^{-1}(0) \), the condition

\[ p_1^\prime(\rho) + \text{Tr}^+ F p_2(\rho) > 0 \]

is satisfied on \( \Sigma \), and where \( X \in \Psi^1_{phg}(\Omega) \) has principal symbol vanishing on \( \Sigma \).

Applications to the case \( L = \sum_{j=1}^m X_j^2 + X_0 \), where \( X_0, X_j \in \Psi^1_{phg}(\Omega) \), with \( X_j, X \) complex-valued, are given.

1. Introduction

In the paper [8] a very special class of properly supported pseudodifferential operators with characteristics of multiplicity 4 was considered and, for \( P = P^* \in \Psi^4_{phg}(\Omega) \) (\( \Omega \) an open subset of \( \mathbb{R}^n \)) belonging to such a class, necessary and sufficient conditions were given in order for

\[ (Pu, u) \geq -C_K \|u\|_{(1/2)}^2, \quad \forall u \in C^\infty_0(K) \]

to hold, where \( K \) is any compact subset of \( \Omega \), and \( C_K > 0 \).

These conditions were given in terms of conditions on \( p_3^\prime(x, \xi) \), the subprincipal symbol of \( P \) defined by \( p_3^\prime(x, \xi) = p_3(x, \xi) + i(\partial_x, \partial_\xi)p_4(x, \xi)/2 \), and on the second-order part of the Weyl-symbol of \( P \), \( \sigma^\wedge_2(P)(x, \xi) = p_2(x, \xi) + i(\partial_x, \partial_\xi)p_3(x, \xi)/2 - (\partial_x, \partial_\xi)^2 p_4(x, \xi)/8 \), at points of \( \Sigma \), the characteristic set of the principal symbol of \( P \), where the symbol of \( P \) is \( p(x, \xi) \sim p_4(x, \xi) + p_3(x, \xi) + p_2(x, \xi) + \cdots \).

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In the sequel, given a real quadratic form $Q(v, v)$, $v$ belonging to a real symplectic vector space $(V, \sigma)$, we shall associate with it, as usual, the fundamental matrix by means of the identity

$$Q(v, w) = \sigma(v, Fw), \quad v, w \in V.$$ 

$Tr^+F$ will then denote the positive trace of $F$, namely $\sum_{\mu_j > 0} \mu_j$, with $i\mu_j$ in the spectrum of $F$.

We now briefly recall the main hypotheses of [8]. Let $P = Q^2 + R$, where $Q = Q^* \in \Psi^2_p h\Sigma(\Omega)$, $R = R^* \in \Psi^3_p h\Sigma(\Omega)$ are properly supported and suppose

(i) $q_2 \geq 0$ vanishes on $\Sigma = q_2^{-1}(0)$, supposed a $C^\infty$-manifold on which the symplectic form $\sigma$ of $T^*\Omega$ has constant rank, and the radical of the Hessian of $q_2$ at $\rho$ is $T_\rho\Sigma$, for any $\rho \in \Sigma$ (this condition is classically referred to as transversal ellipticity of $q_2$ with respect $\Sigma$; see [6], page 361).

(ii) $r_3(\rho) = 0$, $dr_3(\rho) = 0$ when $\rho \in \Sigma$.

The hypotheses imply that the Hessians of $q_2/2$ and $r_3/2$ are invariantly defined on $\Sigma$, and that $\text{Ker} F_Q(\rho) = T_\rho\Sigma$, where we denote by $F_Q(\rho)$ and $F_R(\rho)$ the fundamental matrices associated with the Hessians of $q_2/2$ and $r_3/2$, respectively, at the point $\rho \in \Sigma$. It was then required that

(iii) $[F_Q(\rho), F_R(\rho)] = 0$, $\forall \rho \in \Sigma$.

Notice that in this case $\text{Im} F_Q(\rho)^2$ and $\text{Ker} F_Q(\rho)^2$ are symplectic, invariant subspaces for $F_R(\rho)$. (We shall not mention here the last hypothesis that is made in [8] (namely hypothesis (H5) of the quoted paper) since in our applications the set $\Sigma_c$ introduced in [8] will always be empty.)

Note that in this case one has (with $q_1^t$ denoting the subprincipal symbol of $Q$), for any $\rho \in \Sigma$,

$$F^t_{p}\rho_3(\rho) = 2q_1^t(\rho)F_Q(\rho) + F_R(\rho) \quad \text{and} \quad [F^\rho_3(\rho), F_Q(\rho)] = 0.$$ 

In order to state the result of [8], we give the following definition.

**Definition** Let the fundamental matrices $F_Q$ and $F^\rho_3$ be commuting on $\Sigma$. We say that the quadratic form $\sigma(v, F^\rho_3(\rho)v)$ is not too negative compared to $\sigma(v, F_Q(\rho)v)$, $v \in T_\rho(T^*\Omega)$, $\rho \in \Sigma$, if and only if one of the following occurs:

(a) $s(\rho) \geq -2Tr^+ F_Q(\rho)$,

(b) $s(\rho) = \min_{v \in \left(\text{Ker} F_Q(\rho)^2 / \text{Ker} F_Q(\rho)\right) \setminus \{0\}} \frac{\sigma(v, F^\rho_3(\rho)v)}{\sigma(v, F_Q(\rho)v)}$, 

$$s(\rho) = \min_{v \in \left(\text{Ker} F_Q(\rho)^2 / \text{Ker} F_Q(\rho)\right) \setminus \{0\}} \frac{\sigma(v, F^\rho_3(\rho)v)}{\sigma(v, F_Q(\rho)v)}.$$