Abstract. Suppose \( S \) is a surface of infinite type with pants decomposition \( \Sigma \). We construct a real analytic embedding of an infinite-dimensional parameter space into the Fenchel–Nielsen space of \( S \) with respect to \( \Sigma \), whose image is made up of topologically conjugate Fuchsian groups for which no two groups in the image are quasiconformally conjugate. Moreover, all of the Fuchsian groups in this parameter space have the same Fenchel–Nielsen twist parameters. As a consequence, arbitrarily close (in the Fenchel–Nielsen topology) to a hyperbolic structure for \( S \) there is an infinite-dimensional set of disjoint Teichmüller spaces.

1. Introduction

A major theme in the classical theory of Riemann surfaces is the study of isomorphisms between Fuchsian groups (discrete subgroups of Möbius transformations which keep invariant the unit disc). Since isomorphisms only preserve algebraic information, additional structure is usually required. For instance, the isomorphism may be type preserving, preserve the axes crossed property, or be induced by a self-mapping of the unit disc (see [T], [Mar1]). The focus of this paper is on isomorphisms induced by self-mappings of the closed unit disc. Maps that have continuous extension to the unit circle are essential if the isomorphism is to preserve the dynamical behavior of the group on the boundary circle. Three important and fundamental types of self-maps of the closed unit disc are the (orientation preserving) topological, quasiconformal, and quasi-isometric (with respect to the Poincaré metric) mappings. Of course, if the isomorphism is induced by a Möbius transformation, then we think of these groups as the same. In general, an isomorphism between two Fuchsian groups is induced by a quasiconformal mapping if and only if it is induced by a quasi-isometry (see [D-E]).

For finitely generated Fuchsian groups (that is, finite orbifold type quotient), an isomorphism is induced by a topological mapping of the closed unit disc if and only if it is induced by a quasiconformal mapping (cf. [Mar2]). This is easy to see in the case of co-compact Fuchsian groups since topological equivalence is the same as diffeomorphic equivalence on the quotient Riemann surface, and
a diffeomorphism between compact surfaces is a quasiconformal mapping which lifts to a quasiconformal mapping of the closed unit disc.

The situation for infinitely generated Fuchsian groups (that is, when the quotient surface has infinite orbifold type) is radically different. In stark contrast to the finite case, we produce an infinite-dimensional parameter space of isomorphic groups each induced by a topological mapping of the closed unit disc but not by a quasiconformal mapping. Moreover, between any two of the groups in the parameter space there is no isomorphism induced by a quasiconformal mapping. This parameter space real analytically embeds in the Fenchel–Nielsen space of the orbifold quotient.

2. Flute surfaces with prescribed boundary data and Fenchel–Nielsen coordinates

The unit disc with the Poincaré metric is a model for the hyperbolic plane $\mathbb{H}^2$. The hyperbolic plane together with its boundary circle is denoted $\mathbb{H}^2$ and $\text{PSL}(2, \mathbb{R})$ may be identified with the orientation preserving isometries of $\mathbb{H}^2$. As a reference for the material on Riemann surfaces and Fuchsian groups, we use [Be] and [Mas].

Let $G$ be a Fuchsian group (a discrete subgroup of $\text{PSL}(2, \mathbb{R})$). Denote the Nielsen convex region ([Mas], page 106) of a Fuchsian group $G$ by $\mathcal{N}(G)$. We say that a discrete faithful representation $\rho : G \to \text{PSL}(2, \mathbb{R})$ is a topological (quasiconformal) conjugacy if there is an orientation preserving topological (quasiiconformal) mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$ so that $\rho(g)(z) = fgf^{-1}(z)$ for all $g \in G$ and all $z \in \mathbb{H}^2$. The groups $G$ and $\rho(G)$ are said to be topologically (quasiconformally) conjugate. For the basics on quasiconformal mappings the reader is referred to either [Ah] or [G]. We denote by $\mathbb{R}^\infty$ the set of infinite sequences of real numbers with the product topology. A Fuchsian group $G$ is said to be a pair of pants of orbifold type $(m, n)$, if $\mathbb{H}^2/G$ is a sphere with $m$ points or discs removed and $n$ orbifold points, where $n + m = 3$. Such an orbifold is uniquely determined by the lengths of the $m$ ($m = 0, 1, 2, 3$) boundary geodesics and the orders of the orbifold points. Here we use the convention that length 0 means a point has been removed.

An orbifold pants decomposition $\Sigma$ of the orbifold $S$ is a decomposition of $S = \mathbb{H}^2/G$ into pairs of pants together with a choice of orientation for each boundary geodesic, puncture, or orbifold point on a pair of pants in the decomposition. If $S$ has no orbifold points then $\Sigma$ is a pants decomposition in the usual sense. The homotopy class of a boundary curve in a pair of pants of $\Sigma$ is called a defining curve for $\Sigma$. Let $\text{FN}(S)$ be the space of Fenchel–Nielsen coordinates on $S$ with respect to $\Sigma$. That is, a point in $\text{FN}(S)$ is a marked sequence of lengths of each geodesic which is a boundary geodesic in a pair of pants of $\Sigma$, and a twist parameter for each such geodesic (the sign of the twist parameter is determined by the orientation of the boundary curve). Hence, if $\mathbb{H}^2/G$ is of infinite orbifold type (that is, $G$ is