SHARP FORMS OF NEVANLINNA’S ERROR TERMS

By

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Abstract. Let \( f(z) \) be a meromorphic function in the plane. If \( \psi(t)/t \) and \( p(t) \) are two positive, continuous and non-decreasing functions on \([1, \infty)\) with \( \int_1^\infty dt/\psi(t) = \infty \) and \( \int_1^\infty dt/p(t) = \infty \), then

\[
S(r,f) \leq \log + \frac{\psi(T(r,f))}{p(r)} + O(1)
\]

as \( r \to \infty \) outside a small exceptional set, provided that the divergence of the integral \( \int_1^\infty dt/\psi(t) \) is slow enough. The same forms for the logarithmic derivative and for the ramification term are obtained. It is shown by example that the estimates are best possible.

1. Introduction

In recent years, analogies between value distribution theory and diophantine approximation in number theory have been observed by Lang, Osgood, Vojta and others ([5], [6], [9], [10]). Lang [6] has proposed to find the best error term in Nevanlinna’s second main theorem; this would correspond to the type of numbers in the theory of heights (see [3]–[7], [11] and [13] for earlier work). Let \( f(z) \) be a meromorphic function in the plane. We shall use the usual notation of value distribution theory (see [2], [8] or the up-to-date works [6] and [12]). Nevanlinna’s second main theorem asserts that for any distinct finite complex numbers \( a_1, \ldots, a_q \),

\[
S(r,f) = m(r,f) + \sum_{i=1}^{q} m(r, a_i, f) + N_1(r) - 2T(r,f)
\]

satisfies \( S(r,f) \leq O(\log T(r,f) + \log r) \) as \( r \to \infty \) outside a set of finite linear measure, where

\[
N_1(r) = N\left(r, \frac{1}{f'}\right) + 2N(r,f) - N(r,f')
\]

is the ramification term of the function \( f \).

Wong [11] first made progress in the estimation of \( S(r,f) \). Lang [6] improved Wong’s methods and proved stronger bounds for \( S(r,f), m(r, f'/f) \) and

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$N_1(r)$ involving a positive increasing function $\phi(t) = th(t)$ defined for $t \geq 1$ with $\int_1^\infty dt/\phi(t) < \infty$. Ye [13] has shown that Lang's results are equivalent to the following simpler forms:

\begin{equation}
S(r,f) \leq \log \phi(T(r,f)) + \frac{1}{2} \log h(r),
\end{equation}

\begin{equation}
m\left(r, \frac{f'}{f}\right) \leq \log \phi(T(r,f)) + \frac{1}{2} \log h(r),
\end{equation}

and

\begin{equation}
N_1(r) - 2T(r,f) \leq \frac{1}{2} \log T(r,f) + \log h(T(r,f)) + \frac{1}{2} \log h(r),
\end{equation}

for all $r$ outside a set of finite linear measure.

Then Hinkkanen [3] proved that the term $\log h(r)$ can be completely ignored in (1.3) and (1.4). Examples were first constructed by Miles [7] and also used in [13] and [3] to show the sharpness of (1.3) and (1.4) respectively, in the sense that for any positive non-decreasing and continuous function $h_1(t)$ and $\psi(t) = th_1(t)$ with $\int_1^\infty dt/\psi(t) = \infty$, there is an entire function $f$ such that

\begin{equation}
S(r,f) \geq \log \psi(T(r,f))
\end{equation}

and

\begin{equation}
m\left(r, \frac{f'}{f}\right) \geq \log \psi(T(r,f))
\end{equation}

for all $r$ outside a set of finite measure. By an ingenious example, Ye [13] also showed that (1.5) is essentially best possible in the same sense.

However, there is still a big gap between the upper bounds in (1.3)–(1.5) and the lower bounds in (1.6), (1.7). The purpose of the present paper is to show that for all meromorphic functions the convergent $\phi$ in (1.3), (1.4) and (1.5) can always be replaced by a divergent $\psi$ if the divergence of the integral $\int_1^\infty dt/\psi(t)$ is slow enough. The estimates are best possible and examples will be provided to show that the new upper bounds and the lower bounds differ by constants, at least for many of the functions $p(r)$ considered in (2.2).

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