STRONG ORBIT REALIZATION
FOR MINIMAL HOMEOMORPHISMS

By
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Abstract. Let X be a Cantor set, S a minimal self-homeomorphism of X, and \( \mu \) an S-invariant Borel probability. Let T be an ergodic automorphism of a non-atomic Lebesgue probability space \((Y, \nu)\). Then there is a minimal homeomorphism \( S' \) with the same orbits as S such that \((S', \mu)\) is measurably conjugate to \((T, \nu)\). Here \( S' \) can be chosen strongly orbit equivalent to S if and only if the periodic spectrum of S is contained in the discrete spectrum of T. Corollaries of these results generalize Dye's Theorem and the Jewett–Krieger Theorem.

0. Introduction

In this paper we give Jewett–Krieger type results concerning the realization of an ergodic dynamical system as a minimal homeomorphism of the Cantor set within a given topological orbit equivalence class. These results follow three lines of research in dynamics.

Firstly, they follow a history of realization theorems. By a topological realization of an automorphism T of a probability space \((Y, \nu, B)\) we mean a homeomorphism S of a compact metric space X along with an S-invariant Borel probability \( \mu \) such that \((S, \mu)\) is measurably conjugate to \((T, \nu)\). One of the best known realization theorems is the Jewett–Krieger Theorem: any ergodic automorphism of a non-atomic Lebesgue probability space has a uniquely ergodic topological realization ([K1] see [DGS]). Other realization results include Weiss's construction of a universal minimal realization, i.e. a minimal homeomorphism S of the Cantor set such that given any aperiodic automorphism of a Lebesgue probability space \((T, \nu)\), there is a measure \( \mu \) with which S is a topological realization of \((T, \nu)\) [W].

A second line of research we follow is the study of measurable orbit equivalence. By a measurable orbit equivalence between two automorphisms T and T' of measure-spaces Y and Y' we mean a measure-space isomorphism \( f : Y \rightarrow Y' \) which carries T-orbits onto T'-orbits. Dye proved that there is an orbit equivalence between any two ergodic automorphisms of non-atomic Lebesgue probability spaces ([D] see [KW]). For ergodic non-singular actions, Krieger showed that orbit equivalence is classified by the associated von Neumann crossed product factors ([K2, K3] see [KW]). Rudolph and Kammeyer have assigned a “size” to various restricted orbit equivalences for ergodic actions of discrete amenable groups and
extended Ornstein's classification of finitely determined systems, establishing an analogue for every size [KR, R].

These beautiful results in measurable orbit equivalence have prompted interest in a third line of research, the study of topological orbit equivalence. A topological orbit equivalence between two self-homeomorphisms of compact metric spaces is an orbit-preserving homeomorphism between the spaces. Recently, the remarkable work of Herman, Giordano, Putnam and Skau has shed new light on topological orbit equivalence. Herman, Putnam and Skau associated to each minimal homeomorphism of the Cantor set a unital ordered group [P, HPS]. (This unital ordered group is isomorphic to the first Čech cohomology group of the standard suspension space along with a natural positive cone and order unit [BH2].) Giordano, Putnam, and Skau showed that a quotient of this unital ordered group completely classifies minimal homeomorphisms of the Cantor set up to orbit equivalence ([GPS] see also [GW]). They go on to show that two minimal homeomorphisms have isomorphic unital ordered cohomology groups if and only if they are strongly orbit equivalent.

Two homeomorphisms S and S' of a compact metric space X are strongly orbit equivalent if there is a homeomorphism h : X \rightarrow X and maps m, n : X \rightarrow \mathbb{Z} such that

\[ hS^m(x) = S'h(x), hS(x) = (S')^{n(x)}h(x) \]

and m and n have no more than one point of discontinuity each. While this condition on the cocycles may seem to be very restrictive, the results here (and in [BH1]) show that the strong orbit equivalence class is rich in dynamical systems.

Let (Y, T, \nu) be an ergodic automorphism of a non-atomic Lebesgue probability space, let (X, S) be a minimal homeomorphism of the Cantor set, and let \mu be an ergodic S-invariant Borel probability. In one of our two main results, the Strong Orbit Realization Theorem (SORT, Theorem 2.5), we show that there is a topological realization (S', \mu') of (T, \nu) where S' is a minimal homeomorphism of X strongly orbit equivalent to S if and only if the finite rotations which are (topological) factors of S are (measurable) factors of T. The statement of SORT is a modification of the conjecture of Boyle and Handelman that such a realization (S', \mu') exists for any S and T [BH1]. We present Host's proof of a result of Giordano, Putnam and Skau (Theorem 2.2) which shows that the condition on the finite rotation factors is necessary for strong orbit equivalence [GPS, Ho]. We prove that this condition is sufficient and moreover, we show that the realization (S', \mu') may be chosen such that \mu' is equal to the given S-invariant ergodic measure \mu. For the second main result of this paper, the Orbit Realization Theorem (ORT, Theorem 7.2), we show that regardless of the factors of S and T, there is a minimal homeomorphism S' which is orbit equivalent to S such that (S', \mu) is a topological realization of (T, \nu).

As a corollary of SORT we get a topological version of Dye's Theorem: any two ergodic automorphisms of non-atomic Lebesgue probability spaces have to-