A SUPERREFLEXIVE BANACH SPACE $X$
WITH $L(X)$ ADMITTING A HOMOMORPHISM
ONTO THE BANACH ALGEBRA $C(\beta N)$

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ABSTRACT

A separable superreflexive Banach space $X$ is constructed such that the Banach algebra $L(X)$ of all continuous endomorphisms of $X$ admits a continuous homomorphism onto the Banach algebra $C(\beta N)$ of all scalar valued functions on the Stone–Čech compactification of the positive integers with supremum norm. In particular: (i) the cardinality of the set of all linear multiplicative functionals on $L(X)$ is equal to $2^\omega$ and (ii) $X$ is not isomorphic to any finite Cartesian power of any Banach space.

1. Introduction

The first results showing the existence of a nontrivial linear multiplicative functional acting on the Banach algebra $L(X)$ of all continuous linear endomorphisms of some space $X$ were obtained by B. S. Mityagin and I. C. Edelstein in [10]. Namely, they proved the existence of such a functional acting on the Banach algebra $L(J)$, where $J$ is the well known space constructed by R. C. James in [5] and on the Banach algebra $L(C(\Gamma_{\omega_1}))$, where $C(\Gamma_{\omega_1})$ is the space of all continuous scalar valued functions on the set of ordinals not exceeding the first uncountable ordinal with its usual order topology, equipped with the supremum norm (cf. [12]). A generalization of this result was given by A. Wilansky in [18]. Recently, another construction of a Banach space $X$ with $L(X)$ admitting a nontrivial linear multiplicative functional was given by S. Shelah and J. Steprans [13]. No examples of that kind were known with the underlying Banach space being reflexive. A very simple and well known

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argument yields that the existence of a nontrivial linear multiplicative functional on the Banach algebra of all continuous endomorphisms of a Banach space $X$ implies that $X$ is not isomorphic to any finite Cartesian power of any Banach space (cf. Remark 6.4). Some examples of superreflexive Banach spaces not isomorphic to their Cartesian squares have been known, [2], while an example of a real superreflexive Banach space not isomorphic to the Cartesian square of any Banach space was constructed in [15]. The problem of constructing a complex variant of the example above and the existence of a superreflexive Banach space not isomorphic to any finite Cartesian power of any Banach space were still open (cf. [15], Problems 7.4 and 7.6).

It seems that the main obstacle in constructing a nontrivial linear multiplicative functional on a Banach algebra $L(X)$ lies in the fact that $L(X)$ is "strongly noncommutative". Let us mention, by the way, that no examples were known with $L(X)$ admitting more than one such functional. The problem discussed here motivates the following natural generalizations:

**PROBLEM A.** Does there exist a Banach space $X$ with $L(X)$ admitting a Banach algebra continuous homomorphism onto a "relatively large" commutative Banach algebra $\mathcal{B}$.

Note that the results of [10] and [18] imply the existence of such a homomorphism for some Banach spaces onto a one-dimensional Banach algebra.

**PROBLEM B.** The same as in Problem A but with $X$ being reflexive or even superreflexive.

The aim of this note is to prove the following

**THEOREM 1.1.** There exists a separable superreflexive Banach space $Y$ with the properties:

(i) $Y$ has a finite dimensional decomposition,
(ii) $L(Y)$ admits a continuous homomorphism onto the Banach algebra $C(\beta \mathbb{N})$,
(iii) for every $t \in \mathbb{R}$ there is a projection $P_t \in L(Y)$ and a linear multiplicative functional $\varphi_t$ on $L(Y)$ such that for every $t_1, t_2 \in \mathbb{R}$

$$\varphi_t(P_t) = \begin{cases} 1 & \text{for } t_1 = t_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $C(\beta \mathbb{N})$ denotes the Banach algebra of all continuous scalar valued