CONFORMALITY OF
A QUASICONFORMAL MAPPING AT A POINT

By

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Abstract. Let $f$ be a quasiconformal mapping with the complex dilatation $\mu$. A new condition on $\mu$ is introduced for the conformality of $f$ at a point $z$. The result extends the classical Teichmüller--Wittich--Belinskii regularity theorem.

1 Introduction and main theorems

Let $f(z), f(0) = 0$, be a $K$-quasiconformal homeomorphism of the unit disk $B$ onto itself with the complex dilatation $\mu(z) = f_z/f_z$. The mapping $f$ is conformal at $z = 0$ if $f$ has a non-zero complex derivative at 0, i.e., if

$$\lim_{z \to 0} (f(z)/z) = C, \quad C \neq 0.$$  

Theorem 1.1. If

$$\int_B \frac{|\mu|^2}{|z|^2} \, dA < \infty$$

and if the singular integral

$$\int_B \frac{\mu}{1 - |\mu|^2} \, \frac{dA}{z^2}$$

exists in the sense of the principal value, then $f(z)$ is conformal at $z = 0$.

It is not difficult to see that for quasiconformal maps satisfying (1.2), condition (1.3) is equivalent to the simpler condition that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon < |z| < 1} \frac{\mu}{z^2} \, dA$$

exists. However, we prefer (1.3); see Remark 2.27 below.

Theorem 1.1 is an extension of the following, now classical conformal differentiability theorem, the "TWB-conformality theorem".
Theorem 1.5 (TWB-conformality theorem). If
\begin{equation}
\int_B \frac{|\mu|}{|z|^2} dA < \infty,
\end{equation}
then $f(z)$ is conformal at $z = 0$.

The proof of Theorem 1.5 has a long history. Teichmüller [9] first showed that \eqref{1.6} implies that $|f(z)| \sim C|z|$ as $z \to 0$ for diffeomorphisms $f$ and Wittich [10] extended this to quasiconformal maps. Belinskii [2], [3] and Lehto [6] derived the conformal differentiability from the condition \eqref{1.6}. Reich and Walczak [8] and Brakalova and Jenkins [4] have subsequently given more precise estimates for $|f(z)|$. However, the main difficulty in the proof of Theorem 1.1 concerns $\arg f(z)$.

Simple estimates show that the condition \eqref{1.6} implies the convergence of the integral \eqref{1.2} and the singular integral \eqref{1.3}. The converse does not hold. Moreover, the condition \eqref{1.2} alone is not sufficient for conformality. The bi-lipschitz mapping
\begin{equation*}
f(z) = ze^{i\log \log |z|}, \quad z \in B,
\end{equation*}
which is not conformal at 0, has the dilatation
\begin{equation*}
\mu(z) = \frac{1}{1 + 2i \log |z|} \frac{z}{\overline{z}}
\end{equation*}
and $\mu$ satisfies \eqref{1.2} and does not satisfy \eqref{1.3}. The following result shows to which extent the condition \eqref{1.2} alone is sufficient for conformality.

**Theorem 1.7.** For each measurable function $\mu(z)$ satisfying \eqref{1.2} in $B$ with $||\mu||_\infty < 1$ there exists a quasiconformal mapping $f : B \to B$, $f(0) = 0$, conformal at $z = 0$ and such that the dilatation $\mu_f$ of $f$ satisfies
\begin{equation}
|\mu_f(z)| = |\mu(z)|
\end{equation}
for almost all $z \in B$.

Sharp estimates for $|f'(0)|$ and $\arg f'(0)$ in terms of the integrals \eqref{1.2} and \eqref{1.3} in the case when $f(z)$ is conformal at $z = 0$ are given in Remark 3.14

2 Auxiliary results

We employ the angular dilatation $D_\mu(z)$ of $f(z)$ at $z = 0$ defined as
\begin{equation}
D_\mu(z) = \frac{|1 - \mu(z)\overline{z}/z|^2}{1 - |\mu(z)|^2};
\end{equation}