MEASURES OF TRANSVERSE PATHS IN SUB-RIEMANNIAN GEOMETRY

By

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Abstract. We define a class of lengths of paths in a sub-Riemannian manifold. It includes the length of horizontal paths but also measures the length of transverse paths. It is obtained by integrating an infinitesimal measure which generalizes the norm on the tangent space. This requires the definition and the study of the metric tangent space (in Gromov's sense). As an example, we compute those measures in the case of contact sub-Riemannian manifolds.

1 Introduction

Let \((M, D, g)\) be a \(C^\infty\) sub-Riemannian manifold: \(M\) is a \(C^\infty\) manifold, \(D \subset TM\) a \(C^\infty\) distribution on \(M\) and \(g\) a \(C^\infty\) Riemannian metric on \(D\) (such manifolds are also called Carnot–Carathéodory spaces).

We assume that Chow’s Condition is satisfied: let \(D^s\) denote the \(\mathbb{R}\)-linear span of brackets of degree \(\leq s\) of vector fields tangent to \(D^1 = D\); then, at every \(p \in M\), there exists an integer \(r = r(p)\) such that \(D^r(p) = T_pM\).

Horizontal paths are those paths which are always tangent to \(D\). The length of horizontal paths is then obtained as in Riemannian geometry by integrating the norm of their tangent vectors. Chow’s condition implies that one can join any two points of the manifold by a horizontal path, and therefore a distance \(d\) can be defined.

The purpose of this paper is to introduce a length or a measure for a transverse path, that is, a path whose tangent vector does not belong to the distribution. In [7] and [8], we consider and estimate global measures of paths, such as entropy, Hausdorff measure and complexity. Here we want a definition modeled after the definition of the length: the measure should appear as the integral of an infinitesimal measure, which generalizes the norm of the tangent vector. Defining such an infinitesimal measure of transverse directions requires a careful study of the metric structure of the tangent space.
The notion of metric tangent space, called tangent cone here, has been defined by Gromov [4]. In sub-Riemannian geometry, the tangent cone is itself a sub-Riemannian manifold and has moreover an algebraic structure: a group at a regular point (Mitchell [12]) and a homogeneous space in general (Bellaïche [3]). Recall that a point is regular if all $D^s$, $s \geq 1$, have constant rank around this point. However, these last two papers only describe the tangent cone up to isomorphisms (the privileged coordinates in [3]). Using Mitchell's description, Margulis and Mostow [10, 11] give an intrinsic definition of the tangent cone which holds at regular points. In this work, we need to consider non-regular points. Indeed, to be included in the regular locus is not a generic property of paths (though a point is generically regular).

In Section 2, following the ideas of Margulis–Mostow on one hand and of Bellaïche on the other, we propose an intrinsic definition of the tangent cone at any point. Furthermore, we introduce a filtration of the tangent cone and relate it to the natural filtration of the tangent space defined by the filtration.

In Section 3, we define a class of lengths of paths obtained as an integration of infinitesimal measures on the tangent cone. One of those lengths coincides with the usual length.

Finally, in Section 4 we compute the measures for contact and Martinet sub-Riemannian manifolds, the latter being an important example of non-regular sub-Riemannian manifold (see [1, 13]).

2 Tangent cone

2.1 Privileged coordinates In this section, we recall some results about sub-Riemannian geometry. For a general introduction with references, see [3].

Theorem 1 (Ball-Box Theorem [3], [5]). There exist integers $1 = w_1 \leq \cdots \leq w_n = r$, positive constants $K_1, K_2$ and a system of $C^\infty$ local coordinates $(x_1, \ldots, x_n)$ centered at $p$ (called privileged coordinates at $p$) such that, for any $x$ in $M$ close enough to $p$,

$$K_1 \left( |x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n} \right) \leq d(p, x) \leq K_2 \left( |x_1|^{1/w_1} + \cdots + |x_n|^{1/w_n} \right).$$

Moreover, the privileged coordinates are adapted to the filtration

$$\{0\} \subset D^1(p) \subset D^2(p) \subset \cdots \subset D^r(p) = T_p M$$

in the sense that $dx_j$ vanishes on $D^{w_j-1}(p)$ and does not vanish identically on $D^{w_j}(p)$. 