HIGHER ORDER COMMUTATORS
IN THE REAL METHOD OF INTERPOLATION

By
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Abstract. We extend the real method commutar theorem of Jawerth, Rochberg and Weiss [15] to higher order commutators, thus providing a counter-part to Rochberg's recent results on higher order commutator theorems for the complex method (cf. [27]).

1. Introduction

Rochberg and Weiss [28], and Jawerth, Rochberg and Weiss [15] have shown that, associated with the classical methods of interpolation, there are certain operators, $\Omega$, generally unbounded and non-linear, such that if $T$ is a bounded operator in an interpolation scale, then the commutator $[\Omega, T] = \Omega T - T \Omega$ is also bounded in the scale. This boundedness is due to cancellation since each of the individual terms of the commutator is unbounded. These results have applications in analysis and, in particular, they seem to play an interesting rôle in the study of compensated compactness à la Murat and Tartar as developed in [5]. For example, for a pair of weighted $L^p$ spaces $(L^p(\omega_0), L^p(\omega_1))$ a possible choice for $\Omega$ is $\Omega f = f \log(\omega_1/\omega_0)$; this fact combined with the theory of $A_p$ weights gives the commutator theorem of Coifman–Rochberg–Weiss [6]. In the last few years the theory has been developed in different directions by a number of authors (cf. [7], [8], [16], [17], [22], [26]). For the connection of compensated compactness to interpolation theory we refer to [11], [12], [21], [23], [24], and the references therein.

More recently Rochberg [27] has obtained higher order commutator estimates for the complex method of interpolation. A different approach was found independently in [3]. The purpose of this paper is to develop higher order commutator estimates for the real method of interpolation and thus answer a question raised by Rochberg in [27]. It is expected that our results could have applications to compensated compactness and could, in particular, single out, from an abstract point of view, expressions which exhibit "higher integrability" due to cancellation. In the context of the $L^p$ spaces a typical result, for $n = 2$, can be stated as follows. Suppose that $T$ is a bounded operator in $L^1$ and $L^\infty$, then, for $1 < p < \infty$, there
exist absolute constants $c_p$ such that

$$\| \frac{1}{2} \{ T(\log |f|)^2 - T|f|^2 \} - (T(\log |f|) - T|f|)(\log |Tf|)^2 \|_{L^p} \leq c_p \| f \|_{L^p}. \quad (1)$$

The expressions that can be controlled for large $n$ become progressively more complicated and its individual terms more unbounded, but their combination is bounded due to cancellation. One feature of the abstract methods developed here is that the rôle of the cancellations becomes apparent. Moreover, the real method seems to offer more flexibility and allows us to obtain new commutator results even in the setting of $L^p$ spaces. For example, using Theorem 5 below, we complement (1) with

$$\| T((\log |f|)^n) - (T|f|)^n \|_{L^p} \leq c_p \| f \|_{L^p(L^p)}^{(n-1)} \quad (2)$$

where $r_f$ is the rank function (cf. Section 6 below).

As is the case for the first order commutator theorems, the higher order versions for the real and complex methods are apparently based on different ideas. It is an interesting open problem to develop a unified approach to these results. In this connection it is possible that the methods of twisted sums spaces (cf. [27] and [8]) could be useful.

In the last section of the paper we develop some examples in more detail and we briefly illustrate our results indicating a real variable approach to results in [6], [27], and [1] on higher order commutators of singular integral operators and multiplications with $BMO$ functions. We intend to return to other applications elsewhere. In this connection we mention that our methods could be used to derive higher order versions of the recent commutator results by Iwaniec–Sbordone (cf. [12], [21] and [9]).

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