ON THE LOCALIZATION OF BINDING
FOR SCHRODINGER OPERATORS AND
ITS EXTENSION TO ELLIPTIC OPERATORS

By

YEHUDA PINCHOVER*

Abstract. In this paper we study the asymptotic behavior of the ground state energy \( E(R) \) of the Schrödinger operator

\[ P_R = -\Delta + V_1(x) + V_2(x - R), \quad x, R \in \mathbb{R}^n, \]

where the potentials \( V_i \) are small perturbations of the Laplacian in \( \mathbb{R}^n, n \geq 3 \). The methods presented here apply also in the investigation of the ground state energy \( E(g) \) of the operator

\[ P_g = P + V_1(x) + V_2(gx), \quad x \in X, g \in G, \]

where \( P_g \) is an elliptic operator which is defined on a noncompact manifold \( X \), \( G \) is a discrete group acting on \( X \) by diffeomorphisms \( G \times X \ni (g, x) \mapsto gx \in X \), and \( P \) is a \( G \)-invariant elliptic operator which is subcritical in \( X \).

1. Introduction

In this paper we study the ground state energy \( E(R) \) of the Schrödinger operator

\( P_R = -\Delta + V_1(x) + V_2(x - R), \quad x, R \in \mathbb{R}^n, \)

where the potentials \( V_i \) are small perturbations of the Laplacian in \( \mathbb{R}^n, n \geq 3 \) and the operators \( P_i = -\Delta + V_i(x) \) are nonnegative. Recall [1] that a Schrödinger operator \( P \) defined in a domain \( \Omega \subseteq \mathbb{R}^n \) is nonnegative if and only if there exists a positive function \( u \) which is a solution of the equation \( Pu = 0 \) in \( \Omega \).

Definition 1.1 Let \( P \) be a second order elliptic operator defined in a domain \( \Omega \subseteq \mathbb{R}^n \). A function \( u \) is said to be a ground state in the sense of Agmon of the operator \( P \) in the domain \( \Omega \) with an eigenvalue zero if:

(i) \( u \) is a positive solution of the equation \( Pu = 0 \) in \( \Omega \).

(ii) If \( v \) is any positive solution of the equation \( Pu = 0 \) in some subdomain \( \Omega_1 = \Omega \setminus K_1, K_1 \subset \subset \Omega \) then there exist a positive constant \( C \) and a subdomain \( \Omega_2 = \Omega \setminus K_2, K_2 \subset \subset \Omega \) such that \( u \leq Cv \) in \( \Omega_2 \).

* Harry Goldman Academic Lecturer. This research was partially supported by The Israel Science Foundation administered by The Israel Academy of Sciences and Humanities, by the fund for the promotion of research at the Technion and by the Technion V.P.R. Fund 100-923.

In other words, a ground state is a positive global solution of the equation $Pu = 0$ which has minimal growth at a neighborhood of infinity in $\Omega$.

It is well known that if the operator $P$ admits a positive solution in $\Omega$ then either $P$ admits a minimal (Dirichlet) Green function $G_D^\Omega(x,y)$ and in this case we say that $P$ is subcritical in $\Omega$, or $P$ admits a ground state in the sense of Agmon with an eigenvalue zero and in this case $P$ is said to be critical in $\Omega$. (For more details see Section 2 and also [7, 8].)

The motivation for studying the ground state energy $E(R)$ comes from a remarkable phenomenon known as the Efimov effect for a three-body Schrödinger operator. Such an operator $H$ takes (in the center-of-mass frame) the following form:

$$H = H_0 + \sum_{1 \leq j < k \leq 3} V_{jk}(r_j - r_k),$$

where the operator $H_0$ (the free Hamiltonian) and the operator $H$ act on the space $L^2(\mathbb{R}^3)$, $V_{jk}$ are short-range potentials in $\mathbb{R}^3$ and $r_j$ denotes the position vector of the $j$-th particle. Suppose that all the three two-body subsystems admit ground states in the sense of Agmon with eigenvalues zero (so, $V_{jk}$ are critical potentials) then by the Efimov effect the three-particle system has an infinite number of negative $L^2$-eigenvalues accumulating to zero. Note that this effect holds true even if each pair potential $V_{jk}$ is a function with compact support.

A variational method for proving the Efimov effect was given in [6] by Yu. N. Ovchinnikov and I. M. Sigal and was generalized by H. Tamura in [11, 12] (see also [3, 10] and related results in [2, 13] and in the references therein). The proof relies on the following:

(i) It is well known that the Schrödinger operator $-\Delta + C(1 + |x|)^{-2}$ on $\mathbb{R}^n$ has an infinite number of negative eigenvalues provided that $C < -1/4$.

(ii) Suppose that the functions $V_i, i = 1, 2$ are short range critical potentials in $\mathbb{R}^3$. Then the ground state energy $E(R)$ of the operator $P_R$ in $\mathbb{R}^3$ satisfies

$$\lim_{R \to \infty} |R|^2 E(R) = -\beta < -1/4. \quad (1.2)$$

(iii) The number of the negative eigenvalues of the three-body Hamiltonian $H$ is not less than the number of the negative eigenvalues of a certain Schrödinger operator on $\mathbb{R}^3$ with a potential the leading order term of which is $E(R)$ (hence, the effective interaction is long-range).

So, the study of the ground state energy $E(R)$ of such a type of perturbation is a natural problem in the spectral theory of Schrödinger operators.