ASYMPTOTIC BEHAVIOR OF EIGENVALUES
OF TWO-PARAMETER NONLINEAR
STURM–LIOUVILLE PROBLEMS

By

TETSUTARO SHIBATA

Abstract. The nonlinear two-parameter Sturm–Liouville problem

\[ u'' + \mu g(u) = \lambda f(u) \]

is studied for \( \mu, \lambda > 0 \). By using Ljusternik–Schnirelman theory on the general
level set developed by Zeidler, we shall show the existence of an \( n \)-th variational
eigenvalue \( \lambda = \lambda_n(\mu) \). Furthermore, for special \( f \) and \( g \), the asymptotic formula
of \( \lambda_1(\mu) \) as \( \mu \to \infty \) is established.

1. Introduction

We consider the following nonlinear two-parameter Sturm–Liouville problem

\[ u''(x) + \mu g(u(x)) = \lambda f(u(x)), \quad x \in I = (0, 1), \]
\[ u(0) = u(1) = 0, \]

where \( \mu, \lambda > 0 \) are parameters.

The purpose of this paper is to study the asymptotic behavior of the variational
eigenvalue \( \lambda = \lambda_n(\mu, \alpha) \) obtained by Ljusternik–Schnirelman (LS) theory on the
general level set \( N_\mu \) due to Zeidler [7], where

\[ N_\mu = \left\{ u \in W_0^{1, 2}(I); \int_0^1 \left( \frac{1}{2} u'(x)^2 + \alpha \right) dx = -\alpha \right\}. \]

Here, \( \alpha > 0 \) is a fixed number.

In order to describe and motivate the results of the present paper, let us briefly
recall the results of linear two-parameter problems. Linear two-parameter problems
are studied by many authors. In particular, Binding and Browne [3] considered the
following linear problem:

\[ u''(x) + \mu s(x)u(x) = \lambda r(x)u(x), \quad x \in I, \]
\[ u(0) = u(1) = 0. \]
Under the suitable conditions on $r$ and $s$, they established the following asymptotic formula of $\lambda_n(\mu)$ as $\mu \to \infty$ by using a Prüfer transformation, where $\lambda_n(\mu)$ is the $n$-th eigenvalue of (1.3) for given $\mu > 0$:

\begin{equation}
\frac{\lambda_n(\mu)}{\mu} \longrightarrow \text{ess sup}_{x \in I} \frac{s(x)}{r(x)}.
\end{equation}

We mention here the difference between our problem (1.1) and linear problem (1.3). One of the most important facts is that since (1.1) is nonlinear, we need an additional parameter $\alpha > 0$ to parameterize $\lambda$, that is, $\lambda = \lambda_n(\mu, \alpha)$ and LS theory on general level set $N_\mu$ seems available for our problem (1.1). As is well known, LS theory can be applied to variational nonlinear eigenvalue problems to construct a sequence $(u_n, \lambda_n)_{n \geq 1}$ of solutions such that every $u_n$ lies on the level set $S_R$ and such that $u_n$ attains the critical value obtained by the minimax method. Usually, the level set $S_R$ is diffeomorphic to the unit sphere $S$ in $W_0^{1,2}(I)$. However, for (1.1) it seems available for us to use general level set $N_\mu$, which is not diffeomorphic to $S$ but, roughly speaking, has the structure of a hyperboloid.

Recently, Shibata [6] considered the problem (1.1) in this framework for special $f$ and $g$, that is, $g(u) = u$ and $f(u) = |u|^{p-1}u$, where $p > 1$, and obtained the following asymptotic formula: As $\mu \to \infty$

\begin{equation}
\lambda_n(\mu, \alpha) = \frac{\mu^{(p+1)/2} - O(\mu^{p/2})}{(2\alpha)^{(p-1)/2}}.
\end{equation}

The methods used in [6] are as follows. Let $(u, \mu, \lambda)$ satisfy (1.1) for $g(u) = u$ and $f(u) = |u|^{p-1}u$. Then by scaling $v = \lambda^{1/(p-1)}u$, we see that $v$ satisfies the following one-parameter nonlinear Sturm–Liouville problem:

\begin{equation}
\begin{align*}
-v''(x) + |v(x)|^{p-1}v(x) &= \mu v(x), & \text{for } x \in I, \\
v(0) &= v(1) = 0.
\end{align*}
\end{equation}

Hence, by the precise investigation of the problem (1.6), the formula (1.5) was obtained.

Motivated by the results (1.4) and (1.5), we shall investigate the asymptotic behavior of $\lambda$ as $\mu \to \infty$ by using LS theory on general level sets.

Firstly, we shall establish the existence result of $\lambda = \lambda_n(\mu, \alpha)$ for general nonlinearity $f$ and $g$. Next, we focus our attention on the special case, that is, $f(u) = |u|^{q-1}u$, $g(u) = |u|^{p-1}u$ and shall establish an asymptotic formula of $\lambda_1(\mu)$ as $\mu \to \infty$, such as (1.5). The methods used here are different from those used to