ERGODIC AVERAGES ON SPHERES

By

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Abstract. Let $U_1, U_2, \ldots, U_n$ denote $n$ commuting ergodic invertible measure preserving flows on a probability space $(X, \Sigma, m)$. Let $S_r$ denote the sphere of radius $r$ in $\mathbb{R}^n$, and $\sigma_r$ the rotationally invariant unit measure on $S_r$. Write $U^t x$ to denote $U_1^{t_1} \cdots U_n^{t_n} x$ where $t = (t_1, \cdots, t_n)$. Define the ergodic averaging operator $S_r f(x) = \int_{S_r} f(U^t x) d\sigma_r(t)$. This paper shows that these averages converge for each $f \in L^p(X), p > n/(n-1), n \geq 3$. This is closely related to the work on differentiation by E. M. Stein, S. Wainger, and others. Because of their work, the necessary maximal inequality transfers quite easily. The difficulty is to show that we have convergence on a dense subspace. This is done with the aid of a maximal variational inequality.

1. Introduction

Let $S_r$ denote the surface of the sphere of radius $r$ in $\mathbb{R}^n$, with center at the origin. Let $\sigma_r$ denote the rotationally invariant unit measure on the sphere $S_r$. In 1976 E. M. Stein [6] showed that for $\varphi \in L^p(\mathbb{R}^n), p > n/(n-1), n \geq 3$,

$$\lim_{r \to 0} \int_{S_r} \varphi(s - u) d\sigma_r(u) = \varphi(s) \quad \text{for a.e. } s \in \mathbb{R}^n.$$ 

It is natural to ask if a similar theorem holds in the ergodic theory setting. In particular, let $U_1, U_2, \ldots, U_n$ denote $n$ commuting ergodic invertible measure preserving flows on a probability space $(X, \Sigma, m)$. Write $U^t x$ to denote $U_1^{t_1} \cdots U_n^{t_n} x$ with $t = (t_1, \ldots, t_n)$. We will assume that the flow satisfies the measurability condition that if $E$ is a measurable subset of $X$, then $\{(x, t) : U^t x \in E\}$ is measurable in the product space $(X \times \mathbb{R}^n)$. Define the operator

$$S_r f(x) = \int_{S_r} f(U^t x) d\sigma_r(t).$$

In Theorem 2.1 below we show that as $r \to \infty$, these averages converge a.e. whenever $f \in L^p(X)$, with $p > n/(n-1), n \geq 3$. The proof involves first a transference of a maximal inequality involving an operator on $L^p(\mathbb{R}^n)$ to a maximal inequality involving an operator in the ergodic theory setting. Next a maximal variational inequality is proved, and this is used to show that these spherical averages are very close to the averages over the interior of the sphere.

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Because we know that the averages over the interior of the sphere converge (see Tempelman [9] for example) we can conclude that the spherical averages converge. The variational inequality is the key new ingredient. It is not needed in the differentiation case because in that setting there is a natural dense class, $C_0^\infty(\mathbb{R}^n)$, for which convergence holds for trivial reasons. In the ergodic theory setting there does not seem to be a natural dense class.

We will use the same letter to denote both an operator on $L^p(\mathbb{R}^n)$ and on $L^p(X)$. It will usually be clear from the context which space the operator is acting, but to help distinguish, we will use $\varphi$ to denote an element of $L^p(\mathbb{R}^n)$ and $f$ to denote an element of $L^p(X)$.

We will also need to consider a related averaging operator. Let $B_r$ denote the ball of radius $r$ centered at the origin and define

$$A_r \varphi(s) = \frac{1}{|B_r|} \int_{B_r} \varphi(s-u)du.$$ 

The maximal functions associated with the operators $S_r$ and $A_r$ are defined by

$$S_N^r \varphi(s) = \sup_{r<N}|S_r \varphi(s)|, S^r \varphi(s) = \lim_{N \to \infty} S_N^r \varphi(s), A_N^r \varphi(s) = \sup_{r<N}|A_r \varphi(s)|, A^r \varphi(s) = \lim_{N \to \infty} S_N^r \varphi(s).$$

2. Theorems for flows

The goal of this section is to prove Theorem 2.1 below. We will first prove Theorem 2.1 in the special case $n > 4$. In this case the proof is easier because the relevant Fourier transforms have better decay at infinity. The same ideas will then be used, with more complicated estimates, to establish the case $n = 3$. The arguments follow those of Stein [6], Stein and Wainger [7] and Wainger [10]. The survey article [7] is an excellent exposition of the real variable theory that will be used here. That article also discusses the technical problem associated with the fact that we are working with a singular measure and with functions that are defined only a.e.

**Theorem 2.1** Let $f \in L^p(X)$ with $p > n/(n-1)$, $n > 3$, then $\lim_{r \to \infty} S_rf(x)$ exists for a.e. $x$, and $\lim_{r \to \infty} S_rf(x) = \int_X f(y)dm(y)$ for a.e. $x$.

The proof of this result will follow from a number of lemmas. We begin with a well known fact first observed by A. P. Calderón [2], that many operators on $L^p(\mathbb{R}^n)$ can be “transferred” to the ergodic theory setting.

**Definition** An operator $T$ on $L^p(\mathbb{R}^n)$ is *semi-local* if there exists a number $N$ such that whenever the support of a function $\varphi$ is contained in a cube with sides $L$, the support of $T(\varphi)$ will be contained in a cube with sides $L + N$. 