AN INHOMOGENEOUS ELLIPTIC COMPLEX

By
NEDA BOKAN, PETER GILKEY* AND RADE ŽIVALJEVIĆ

Abstract. We define a 3 term sequence $\mathcal{P}$ of differential operators of mixed type; the first and third operators are 1st order while the second operator is 2nd order. $\mathcal{P}$ is always elliptic; it forms a complex iff $M$ is Einstein. It was first discussed by Gasqui. $\mathcal{P}$ is related to similar complexes $\mathcal{C}$ and $\mathcal{G}$ discussed by Calabi and Gasqui–Goldschmidt. The index and equivariant index of $\mathcal{P}$ vanish. In dimension 2, $\mathcal{P} = \mathcal{C} \oplus \mathcal{S}$ where $\mathcal{S}$ is of Dirac type; $\mathcal{C}$ and $-\mathcal{S}$ determine the same equivariant index. We study the heat equation asymptotics of the operators of $\mathcal{P}$; the associated Laplacians do not have scalar leading symbol.

0. Introduction

Let $M$ be a compact Riemannian manifold without boundary of dimension $m \geq 2$. We define the sequence $\mathcal{P}$ on $M$ as follows. Let $T(M)$ be the tangent space of $M$ and let $S^2(M)$ be the bundle of symmetric 2-tensors. If $X \in C^\infty(T(M))$, let $L_X(ds^2)$ be the Lie derivative of the metric. If $h \in C^\infty$, let

$$ds^2(\epsilon) = ds^2 + \epsilon h.$$  

$ds^2(\epsilon)$ is positive definite for small $\epsilon$. Let $\rho(\epsilon)$ be the Ricci curvature of $ds^2(\epsilon)$ and let $\frac{d}{d\epsilon}|_{\epsilon=0}\rho(\epsilon)$ be the linearization. Let $\tau$ be the scalar curvature and let $c = \tau/m$ be the Einstein coefficient; $M$ is Einstein $\iff \rho = c \cdot ds^2$ and $c$ is constant. Let $P_2$ be the contracted second Bianci identity and let $(\rho h + h\rho)_{ij} = \rho_{ik}h_{kj} + h_{ik}\rho_{kj}$. Then $\mathcal{P}$ is the sequence:

$$P_0(X) = L_X(ds^2) : C^\infty(T(M)) \rightarrow C^\infty(S^2(M)),$$

$$P_1(h) = -2\frac{d}{d\epsilon}|_{\epsilon=0}\rho(\epsilon) + \rho h + h\rho : C^\infty(S^2(M)) \rightarrow C^\infty(S^2(M)),$$

$$P_2(h) = (h_{ijj} + h_{jij} - h_{jiji}) : C^\infty(S^2(M)) \rightarrow C^\infty(T(M)).$$

In §1, we will expand the $P_i$ in terms of covariant differentiation and show $\mathcal{P}$ is elliptic; i.e. $\mathcal{P}$ is exact on the symbol level. The normalizing constants are motivated by those formulas.

$\mathcal{P}$ is integrable if $P_2P_1 = P_1P_0 = 0$. We will see $\mathcal{P}$ is integrable $\iff M$ is Einstein; if $m = 2$, Einstein means constant scalar curvature. We assume $M$ is Einstein in §2.

* Research partially supported by the NSF, NSA, IHES, and Ohio State

§3, §4, and §5. Since we can always ‘roll up’ any elliptic complex, the integrability condition is inessential for index theory. $\mathcal{P}$ is natural in the category of Riemannian manifolds. Most elliptic complexes involve first order operators; $\mathcal{P}$ is of particular interest as $P_1$ is second order. We will show in §2 that the index and equivariant index of $\mathcal{P}$ are trivial.

Elliptic sequences with $P_0(X) = L_X(ds^2)$ have been discussed previously by many authors; we acknowledge with pleasure the role of P. Michor in bringing this problem to our attention. Calabi [Ca] defined an elliptic complex $\mathcal{C}$ resolving the sheaf of Killing vector fields in the category of space forms. Gasqui–Goldschmidt [GG] gave an elliptic complex $\mathcal{G}$ resolving the sheaf of conformal Killing vector fields in the category of conformally flat manifolds. Both complexes involved $m + 1$ terms and satisfied the Poincare lemma. $\mathcal{P}$ on the other hand involves only 4 terms and consequently cannot satisfy the Poincare lemma and does not resolve the sheaf of Killing vector fields. However, $\mathcal{P}$ is defined in a wider category of manifolds. $\mathcal{P}$ was first discussed by Gasqui [Ga] from a very different point of view; Gasqui was interested in local resolvability of the Einstein equations while we are interested in the index theory of such a complex. We also refer to Berger and Ebin [BE] for a description of some elliptic operators on $S^2(M)$ and related work by Barbance [B].

The differing integrability conditions arise as follows. The second operator of $\mathcal{C}$ is the full linearized curvature tensor. Since the full curvature tensor is involved, the integrability condition is the most restrictive; $M$ must have constant sectional curvature. The second operator of $\mathcal{G}$ is the projection of the full linearized curvature tensor on the Weyl conformal tensor; this leads to the integrability condition that $M$ is conformally flat. Instead of projecting on the Weyl conformal tensor, we shall project on the orthogonal complement and use the normalized linearized Ricci tensor to define $\mathcal{P}$; it is necessary to adjust by the Einstein constant. The integrability condition is that $M$ is Einstein. Thus $\mathcal{P}$ is in a certain sense orthogonal to $\mathcal{G}$.

In §3, we compute the associated Laplacians and prove some vanishing theorems. We show that if $\rho < 0$, then $H^0(\mathcal{P}) = H^3(\mathcal{P}) = 0$ and $H^1(\mathcal{P}) = H^2(\mathcal{P})$. We will show in §4 that $\mathcal{P}$ is irreducible if $m \geq 3$. If $m = 2$, then $\mathcal{P} = \mathcal{C} \oplus \mathcal{S}$ where $\mathcal{S}$ is a complex of Dirac type. We will use this decomposition to compute the equivariant index of $\mathcal{C}$ in this case. We study several examples on different Riemann surfaces. In §5, we study the associated heat equation invariants. We hope the example provided by $\mathcal{P}$ will prove instructive as $\mathcal{P}$ is very different from the standard examples. We acknowledge with gratitude the assistance of the referee in simplifying several arguments.