MOUNTAIN PASS THEOREM IN ORDER INTERVALS
AND MULTIPLE SOLUTIONS FOR
SEMILINEAR ELLIPTIC DIRICHLET PROBLEMS

By

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Dedicated to P. H. Rabinowitz on the occasion of his 60th birthday

Abstract. In this paper, we prove a mountain pass theorem in order intervals in which the position of the mountain pass point is given precisely in terms of the order structure. By using this result and constructing special flows, we deal with the existence of multiple solutions and sign-changing solutions for the following classes of elliptic Dirichlet boundary value problems: (1) nonlinear terms have concave property near zero and have superlinear but subcritical growth at infinity; (2) nonlinear terms are of the form $h(x)f(u)$, with $h(x)$ changing sign; (3) the asymptotically linear case. We obtain several new existence results of nodal solutions and give more comparable relations among the positive, negative and sign-changing solutions obtained. Our method is set up in an abstract setting and should be useful in other problems.

1 Introduction

This paper is concerned with the existence and nodal property of multiple nodal solutions for several classes of elliptic Dirichlet problems. We shall give new multiplicity results for sign-changing solutions and provide order relations between solutions. In order to achieve these results, we first consider some abstract results and prove a variant of the mountain pass theorem due to Ambrosetti and Rabinowitz [4] in order intervals; our result has the order structure built in and gives the location of the mountain pass point in terms of the ordered intervals. Second, according to the nature of the problems, we construct special flows which possess a special invariance property though the original flow may not even under

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linear perturbations. Using these results, we deal with several classes of nonlinear elliptic Dirichlet problems.

Let us start with the abstract results. Let $E$ be a Hilbert space and $P_E \subset E$ a closed convex cone. Let $X \subset E$ be a Banach space which is densely embedded to $E$. Let $P = X \cap P_E$ and assume that $P$ has nonempty interior $P$. We assume any order interval is finitely bounded. Let $\Phi$ be a functional from $E$ to $\mathbb{R}$ which satisfies the following assumptions.

- $(\Phi_1)$ The function $\Phi$ belongs to $C^2(E, \mathbb{R})$ and satisfies the (PS) condition in $E$ (e.g. $[3]$) and the deformation property in $X$ (e.g. $[7]$); $\Phi$ only has finitely many isolated critical points.

- $(\Phi_2)$ The gradient of $\Phi$ is of the form $\nabla \Phi = Id - K_E$, where $K_E : E \to E$ is compact. $K_E(X) \subset X$ and the restriction $K = K_E|_X : X \to X$ is of class $C^1$ and strongly order preserving, i.e., $u > v \implies K(u) \gg K(v)$ for all $u, v \in X$, where $u \gg v \iff u - v \in \hat{P}$.

- $(\Phi_3)$ $\Phi$ is bounded from below on any order interval in $X$.

These assumptions are quite standard in treating known problems and in developing abstract critical point theory in partially ordered Hilbert spaces ($[8],[12-15],[17]$). However, for the applications in this paper, some of these assumptions may not be satisfied. For example, $\Phi$ is only $C^1$ in some problems; we give a special construction of a pseudo-gradient vector field to overcome this and to have the order structure built in. This will be done in the appendix. In another application, we have to deal with a situation in which $K$ is not strongly order preserving, even after we add a linear strongly order preserving map. To overcome this difficulty, we shall construct a special vector field which may not be a pseudo-gradient vector field. Nevertheless, we show that the flow under this vector field still satisfies the deformation property for the functional involved and possesses a certain prescribed invariance property. This will be done in Section 2, in which we also state some generalizations of the abstract results stated below.

Now we state some abstract results which will be used later in applications to elliptic boundary value problems.

**Theorem 1.1.** Suppose $\Phi$ satisfies $(\Phi_1), (\Phi_2), (\Phi_3)$ and $u < \bar{u}$ is a pair of a subsolution and a supersolution of $\nabla \Phi = 0$ in $X$. Then $[u, \bar{u}]$ is positively invariant under the negative gradient flow of $\Phi$, and $-\nabla \Phi$ points inward in $[u, \bar{u}]$. Moreover, if $u < \bar{u}$ is a pair of a strict subsolution and supersolution of $\nabla \Phi = 0$ in $X$, then

$$\deg(Id - K', [u, \bar{u}], 0) = 1.$$