EXPLICIT REALIZATION OF A METAPLECTIC REPRESENTATION

By

Y. FLICKER,[†] D. KAZHDAN[††] AND G. SAVIN[‡]

0. Let $F \neq \mathbf{C}$ be a local field with *char* $F \neq 2$. In [W] Weil explicitly constructed a model of a genuine unitary representation θ of the two-fold covering group $\tilde{\mathrm{S}}\mathrm{p}$ of the symplectic group Sp over F. In particular, the existence of the covering group $\tilde{\mathrm{S}}\mathrm{p}$ was first proven in [W]. It is now known (see, e.g., [M]) how to construct r-fold covering groups of split semi-simple groups over a field $F \neq \mathbf{C}$ containing a primitive rth root of unity. In particular, when $r = 2$, such F has *char* $F \neq 2$. In the case of GL(n) the analogous genuine unitarizable representation Θ of a covering group is defined in [KP1] as a sub- or quotient of some induced representation. This Θ corresponds to the trivial representation of GL(n) by the metaplectic correspondence (see [KP2], [FK1]). The purpose of this paper is to construct an explicit model of the representation $\Theta = \Theta_3$ of a two-fold covering group G of GL(3) over a local field $F \neq \mathbf{C}$ of characteristic $\neq 2$, analogous to the explicit model of the representation of Weil [W]. We also determine the unitary completion of the unitarizable Θ_3. The unitary completion of our model coincides with the model of Torasso [T] when $F = \mathbf{R}$. The existence of our model has interesting applications in harmonic analysis. Some of these applications are discussed in detail in §3. In a sequel [F1] the techniques of this paper are generalized to construct an explicit model of Θ_n for any $n \geqq 3$.

1. The representation

To state our Theorem and its Corollaries, we begin by specifying the representation Θ to be studied.

1.1. Let F be a local field $\neq \mathbf{C}$ of characteristic $\neq 2$. For every integer $n > 1$ there exists (see [M]) a unique non-trivial topological central double covering group $p : S_n \rightarrow \mathrm{SL}(n, F)$. Choose a section $\mathbf{s} : \mathrm{SL}(n, F) \rightarrow S_n$ corresponding to a choice of a two-cocycle $\beta_n' : S_n \times S_n \rightarrow ker\ p$ which defines the group law on S_n. Embed $\bar{G}_n = \mathrm{GL}(n, F)$ in $\mathrm{SL}(n + 1, F)$ by

[†] Partially supported by an NSF grant and a Seed grant. This author wishes to express his deep gratitude to IHES for its hospitality when the paper was written.
[††] Partially supported by an NSF grant.
[‡] Sloan Doctoral Fellow.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 55 (1990)

$$\iota : g \to \begin{pmatrix} g & 0 \\ 0 & det\, g^{-1} \end{pmatrix}.$$

Denote by G_n' the preimage $p^{-1}(\iota(\bar{G}_n))$. Let $(\cdot, \cdot) : \mathrm{F}^2 \times \mathrm{F}^2 \to \{1, -1\}$ be the Hilbert symbol. Identify $\{1, -1\}$ with the kernel of p. Put $\beta(g, g') = \beta'(g, g')(det\, g, det\, g')$ (g, g' in $\bar{G}_n$). Let $\mathrm{s} : \bar{G}_n \to G_n'$ be the restriction of the section used in the definition of S_{n+1}. Denote by G_n the group which is equal to G_n' as a set, whose product rule is given by $\mathrm{s}(g)\zeta \cdot \mathrm{s}(g')\zeta' = \mathrm{s}(gg')\zeta\zeta'\beta(g, g')$. Then G_n is a non-trivial topological double covering group of $\bar{G}_n$. Let $\bar{A}$ and $\bar{B}$ be the groups of diagonal and upper-triangular matrices in $\bar{G}_n$, and A and B their preimages in G_n. Note that s is a homomorphism on the group $\bar{N}$ of upper-triangular unipotent matrices, and put $N = \mathrm{s}(\bar{N})$. Let $\bar{Z}$ be the center of $\bar{G}_n$ and Z the center of G_n.

Lemma 1. *Let $\bar{A}^2$ be the group of squares in $\bar{A}$, and put $A^2 = p^{-1}(\bar{A}^2)$. Then*
(i) *the group ZA^2 is the center of A,*
(ii) *if n is even then $Z = A^2 \cap p^{-1}(\bar{Z})$,*
(iii) *if n is odd then $Z = p^{-1}(\bar{Z})$, and p defines an isomorphism*

$$p : Z/(Z \cap A^2) \to \bar{Z}/\bar{Z}^2 \cong F^\times/F^{\times 2}.$$

Proof. See [KP1], Prop. 0.1.1.

Define a map $t = t_n : \bar{A} \to A^2$ by $t(h) = \mathrm{s}(h)^2 u(h)$, where

$$u(h) = \Pi_{1 \leqq i < j \leqq n} (h_i, h_j)$$

for a diagonal matrix $h = diag(h_i)$ with entries h_i ($1 \leqq i \leqq n$). Note that t is independent of the choice of the section s. Using the product rule in G_n (see [KP1], p. 39), it is easy to check that our section s satisfies $t(h) = \mathrm{s}(h^2)$ for every h in $\bar{A}$.

Lemma 2. *The map t is a group homomorphism.*

Proof. This follows from the multiplication law on $A \subset G_n$.

Definition. Let $\bar{\delta} = \bar{\delta}_n : \bar{A} \to \mathbf{C}^\times$ be the character $\bar{\delta}(diag(h_i)) = \Pi_{i=1}^n |h_i|^{(2i-1-n)/2}$. A character $\delta = \delta_n : ZA^2 \to \mathbf{C}^\times$ whose restriction to $ker\, p$ is non-trivial is called *exceptional* if $\delta(t(h)) = \bar{\delta}(h)$ for all h in $\bar{A}$.

Note that $A^2 = t(A) \cdot ker\, p$ is equal to ZA^2 if n is even. If n is odd then $ZA^2/A^2 \cong F^\times/F^{\times 2}$, hence it is possible to extend δ from A^2 to ZA^2, and there exist exceptional characters for all n.

Lemma 3. (i) *For any exceptional character δ of ZA^2 there exists a unique (up to isomorphism) irreducible representation ρ_δ of A whose restriction to ZA^2 is* $\delta \cdot \mathrm{Id}$.