On a Solution of the Matter Conversion Equation.

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Summary. – A conceptional simple solution of a system of coupled nonlinear ordinary differential equations describing matter conversion processes is presented and is exemplified for a reduced chain of nuclear reactions.

Matter conversion takes place if in a closed system the nuclear composition is changing with time. Such conversions occur in stars and can be provoked in fusion facilities.

The general equation describing macroscopically such matter conversion of a system, where \( Y_i \) (\( i = 1, 2, 3; \ldots, n \)) nuclei are involved in reactions, reads

\[
\frac{dY_i(t)}{dt} = \sum_{j=1}^{m} b_{ij} \prod_{s=1}^{n} Y^s_{rs}(t),
\]

\( b_{ij} \) are interaction parameters which can depend on time. Assuming two-body interactions only, one has

\[
\sum_{s=1}^{n} y_{ij} = 2,
\]

for any \( i, j \).

The solution of (1) is usually only numerically possible.

We propose here a conceptional simple algorithm for solving eq. (1).

Supposing that \( b_{ij} \) are constants the formal solution of (1) reads

\[
Y_i(t) = \sum_{s=0}^{\infty} \frac{(t-t_o)^s}{s!} \sum_{H,K} (a_{i,H,K} B^H) s \gamma_1 y_2 y_3 \ldots y_n,
\]

where \( H \) is a matrix with the elements \( h_{ij} \),

\[
K = k_1 k_2 k_3 \ldots k_n, \quad i = 1, 2, 3, \ldots, n, \quad j = 1, 2, 3, \ldots, m,
\]

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\( y_i \) are the initial values of \( Y_i \), and \( B^s \) is an abbreviation for
\[
B^s = \prod_{i=1}^{n} \prod_{j=1}^{m} b_{ij}^s.
\]
Note that \( B \) does not depend on \( s \). The exponent \( H \), however, changes with varying \( s \).

The representation of the functions \( Y_i(t) \) consists thus of three main parts: the variable-dependent part \((t - t_0)s/s!\), the coefficients \( a_{i,H,K}B^H \) and the initial-value-dependent part \( y_1^i,y_2^i,y_3^i \ldots y_n^i \).

To compute \( Y_i(t) \) in (2) the coefficients \( a_{i,H,K} \) have to be determined. If they are known, then the labels \( H \) and \( K \) and thus \( B^H \) are also known.

With the aid of mathematical induction one obtains the following general recurrence relation for the \( a_{i,H,K} \):
\[
(a_{i,H,K},k_2,\ldots,k_n)_{s+1} = \left( \sum_{p=1}^{n} \sum_{j=1}^{m} (k_p + 1 - r_{pj})A(pj) \right),
\]
with
\[
A(pj) = a_{i,H,K,pj} + (k_p - r_{pj})A(pj),
\]
where \( H_{pj} = H - I_{pj} \), \( I_{pj} \) is a zero matrix except the element \( pj \) which equals 1. For example \( H_{23} \) means the element \( h_{23} \) becomes \( h_{23} - 1 \) and \( H \) remains unchanged else.

Recurrence relation (3) starts with the values
\[
(a_{i,H,K})_{s=0} = (a_{i=0,H=0,K=0} - \ldots, k_n - 1),
\]
for example, \( (a_{2,0,0}|0,0,0,0)^{s=0} = 1 \).

The following general remarks can be made on the recurrence relation (3):

This relation permits computation of the coefficients of the series term of order \( s + 1 \) from coefficients of the series terms of order \( s \). The indices \( H \) and \( K \) are automatically given by (3).

Symmetries of (1) lead to a reduction of the number of coefficients \( a_{i,H,K} \) to be computed (see example below).

The coefficients \( a_{i,H,K} \) and thus also \( B^H \) have only to be computed once for a given system (1). If only \( B \) is changed—\( B \) contains essentially the physics of the system—, then only \( B^H \) have to be recalculated as \( a_{i,H,K} \) remains unchanged. The integration of (1) for different initial values is, therefore, reduced to the summation of the power series (2). This renders the present algorithm numerically economical.

In certain cases, the recurrence relation (3) generates number sequences which are known from other mathematical disciplines, for example, in combinatorial theory \((1)\).

We exemplify this algorithm for a simple matter conversion process in which three nuclei are involved:
\[
D + T \rightarrow ^4\text{He} + n, \quad D + ^4\text{He} \rightarrow T + ^3\text{He}.
\]