VALUE DISTRIBUTION OF PAINLEVÉ TRANSCENDENTS OF THE FIRST AND THE SECOND KIND

By

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Abstract. We examine value distribution properties of the first and the second Painlevé transcendents. For every transcendental meromorphic solution \( \phi(z) \) (resp. \( \psi(z) \)) of the first (resp. second) Painlevé equation, the deficiency \( \delta(g, \phi) \) (resp. \( \delta(g, \psi) \)) of a small function \( g(z) \) does not exceed \( 1/2 \). Furthermore, for \( \phi(z) \), the ramification index satisfies \( \vartheta(g, \phi) \leq 5/12 \).

1 Introduction

Consider the first and the second Painlevé equations

\[
\begin{align*}
(I) & \quad w'' = 6w^2 + z, \\
(II) & \quad w'' = 2w^3 + zw + \alpha
\end{align*}
\]

\( (= d/dz) \), where \( \alpha \in \mathbb{C} \). All the solutions of these equations are meromorphic in the whole complex plane ([2]). It is easy to see that every solution of (I) is transcendental. Equation (II) admits a unique rational solution, if and only if \( \alpha \in \mathbb{Z} \) ([6]). For transcendental meromorphic solutions of these equations, the deficiency \( \delta(a, w) \) and the ramification index \( \vartheta(a, w) \) \( (a \in \mathbb{C} \cup \{\infty\}) \) were examined in [3], [7], [8], [11]. Throughout this paper, we use the standard notation of value distribution theory for a meromorphic function \( f(z) \), such as \( m(r, f), N(r, f), N_1(r, f), T(r, f), S(r, f), \delta(a, f), \vartheta(a, f) \) \( (r > 0, a \in \mathbb{C} \cup \{\infty\}) \) ([4]). We say that a meromorphic function \( g(z) \) is small with respect to \( f(z) \) if \( T(r, g) = S(r, f) \). Define the deficiency and the ramification index of a small function \( g(z) \) by

\[
\begin{align*}
\delta(g, f) &= \liminf_{r \to \infty} \frac{m(r, 1/(f-g))}{T(r, f)}, \\
\vartheta(g, f) &= \liminf_{r \to \infty} \frac{N_1(r, 1/(f-g))}{T(r, f)}.
\end{align*}
\]
The purpose of this paper is to estimate these quantities for transcendental meromorphic solutions of (I) and (II). Let \( \phi(z) \) (resp. \( \psi(z) \)) be an arbitrary transcendental meromorphic solution of (I) (resp. (II)). Our main results are stated as follows:

**Theorem 1.1.** If \( g(z) \) is small with respect to \( \phi(z) \), then \( \delta(g, \phi) \leq 1/2 \).

**Theorem 1.2.** If \( g(z) \) is small with respect to \( \psi(z) \), then \( \delta(g, \psi) \leq 1/2 \).

**Theorem 1.3.** If \( g(z) \) is small with respect to \( \phi(z) \), then \( \vartheta(g, \phi) \leq 5/12 \). Moreover, for every complex number \( a \in \mathbb{C} \), \( \vartheta(a, \phi) \leq 1/6 \).

Our results are proved in Sections 3, 4 and 5. The auxiliary functions \( H(z) \) and \( K(z) \) in the proofs of Theorems 1.1 and 1.2 are due to [10]. In the proof of Theorem 1.3, we need to choose a different type of auxiliary function \( \Omega(z) \) or \( \Lambda(z) \). In these proofs, we use the following additional notation: for a set \( A \subset \mathbb{C} \),

\[
N(r, f)|_A = \int_0^r \left( n(\rho, f)|_A - n(0, f)|_A \right) \frac{d\rho}{\rho} + n(0, f)|_A \log r,
\]

\[
N_1(r, f)|_A = \int_0^r \left( n_1(\rho, f)|_A - n_1(0, f)|_A \right) \frac{d\rho}{\rho} + n_1(0, f)|_A \log r,
\]

\[
n(\rho, f)|_A = \sum_{\sigma \in A, |\sigma| \leq \rho, f(\sigma) = \infty} \mu(\sigma, f), \quad n_1(\rho, f)|_A = \sum_{\sigma \in A, |\sigma| \leq \rho, f(\sigma) = \infty} (\mu(\sigma, f) - 1),
\]

where \( \mu(\sigma, f) \) denotes the multiplicity of the pole \( z = \sigma \) of \( f(z) \).

## 2 Lemmas

Substituting the Laurent series expansion of the solution \( \phi(z) \) (or \( \psi(z) \)) into (I) (or (II)), we have the following (see, for example, [1; (12.2.5)], [2; (5.1)])).

**Lemma 2.1.** Around a movable pole \( z = z_\infty \) of \( \phi(z) \),

\[
\phi(z) = \frac{1}{(z - z_\infty)^2} - \frac{z_\infty}{10}(z - z_\infty)^2 - \frac{1}{6}(z - z_\infty)^3 + \gamma(z - z_\infty)^4 + O((z - z_\infty)^5);
\]

and, around a movable pole \( z = z_\infty \) of \( \psi(z) \),

\[
\psi(z) = \frac{\pm 1}{z - z_\infty} \mp \frac{z_\infty}{6}(z - z_\infty) + \frac{\mp 1 - \alpha}{4}(z - z_\infty)^2 - \frac{1}{2}(z - z_\infty)^3 + \gamma(z - z_\infty)^4 + O((z - z_\infty)^5).
\]

Here \( \gamma \) is a parameter depending on the initial condition.

Putting \( P(z, u) = u \) in [4; Lemma 2.4.2], we have the following.