NORMAL FAMILIES AND FIXED POINTS

By

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Abstract. Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D$ and let $k \geq 2$ be a positive integer. If, for every $f \in \mathcal{F}$, its $k$-th iterate $f^k$ has no fixed point in $D$, then $\mathcal{F}$ is normal in $D$.

1 Introduction

Let $D$ be a domain in $\mathbb{C}$ and $\mathcal{F}$ a family of meromorphic functions defined in $D$. $\mathcal{F}$ is said to be normal in $D$, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in $D$, to a meromorphic function or $\infty$ (see Hayman [6], Schiff [8], Yang [11]).

Let $f$ be a meromorphic function in a domain $D \subset \mathbb{C}$ and $k$ a positive integer. We say that $z_0 \in D$ is a fixed point of $f^k$ (the $k$-th iterate of $f$) if $f^j(z_0) \in D$ for $j = 1, 2, \ldots, k - 1$ and $f^k(z_0) = z_0$.

Here and in the sequel, $f^1(z) = f(z)$, $f^2(z) = f(f(z))$, and $f^k(z)$ is defined inductively via $f^k(z) = f(f^{k-1}(z))$ for $k = 2, 3, \ldots$ .

If $z_0$ is a fixed point of $f^k$ satisfying $|(f^k)'(z_0)| > 1$, then $z_0$ is called a repelling fixed point of $f^k$.

In 1992, Yang [10, Problem 8] posed the following problem.

Problem 1. Let $\mathcal{F}$ be a family of entire functions, $k \geq 2$ a positive integer and $D$ a domain in $\mathbb{C}$. If, for every $f \in \mathcal{F}$, both $f$ and its $k$-th iterate $f^k$ have no fixed point in $D$, is $\mathcal{F}$ normal in $D$?


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**Theorem A.** Let \( \mathcal{F} \) be a family of holomorphic functions in a domain \( D \). If, for every \( f \in \mathcal{F} \), there exists a positive integer \( k = k(f) \geq 2 \) such that \( f^k \) has no repelling fixed point in \( D \), then \( \mathcal{F} \) is normal in \( D \).

It is natural to pose the following problem.

**Problem 2.** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \) and \( k \geq 2 \) a positive integer. If, for every \( f \in \mathcal{F} \), its \( k \)-th iterate \( f^k \) has no fixed point in \( D \), is \( \mathcal{F} \) normal in \( D \) ?

For Problem 2, Wang and Wu [9] improved a result of Essén and Wu [4] and proved

**Theorem B.** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \) and \( q \) a non-negative integer. If, for every \( f \in \mathcal{F} \), there is a positive integer \( k \geq 2 \) such that the \( k \)-th iterate of \( f \) has at most \( q \) fixed points in \( D \), then \( \mathcal{F} \) is quasinormal with order not greater than \( \max(4, q + 3) \) in \( D \).

For the notion of quasinormal families, see Schiff [8]. In this paper, we give an affirmative answer to Problem 2.

**Theorem 1.** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \) and \( k \geq 2 \) a positive integer. If, for every \( f \in \mathcal{F} \), its \( k \)-th iterate \( f^k \) has no fixed point in \( D \), then \( \mathcal{F} \) is normal in \( D \).

Naturally, we ask that whether "its \( k \)-th iterate \( f^k \) has no fixed point in \( D \)" can be replaced by "its \( k \)-th iterate \( f^k \) has no repelling fixed point in \( D \)" in Theorem 1. The following example shows that the answer is negative.

**Example 1.** Let

\[ \mathcal{F} = \{ f_n(z) = 1/(nz) : n = 1, 2, \ldots \}, \quad D = \{ z : |z| < 1 \}. \]

Then \( \mathcal{F} \) is a family of meromorphic functions in \( D \). Obviously, \( f_n^2 = \text{id} \) for each function \( f_n \in \mathcal{F} \), where \( \text{id} \) denotes the identity function. Hence if \( k \) is even, \( f_n^k(z) = z \); while, for \( k \) odd, \( f_n^k(z) = 1/(nz) \). Thus for any positive integer \( k \) and any \( f_n \in \mathcal{F} \), \( f_n^k \) has no repelling fixed point in \( D \). But \( \mathcal{F} \) is not normal at \( z = 0 \).

2 **Some lemmas**

In order to prove Theorem 1, we require the following results.

**Lemma 1** ([7, Lemma 2], cf. [12, p. 217]). Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( D \), all of whose zeros have multiplicity at least \( k \), and suppose