A NOTE ON "UNIVERSAL" PHRAGMÉN-LINDELÖF THEOREMS AND A LEMMA OF BEURLING

By

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In this note, we combine a lemma of Beurling [2] with some ideas from Toppila [8] to discuss the "universal" Phragmén-Lindelöf conjecture of D. Newman (cf. [3, Problem 7.46]).

**Beurling's lemma.** Let $L_m$, with $1 \leq m \leq p$, be simple Jordan arcs in the complex plane from 0 to $\infty$ with no common points except 0. Let $D_m$, with $1 \leq m \leq p$, be the simply-connected domains bounded by the curves. For every $m$, $u_m$ is assumed to be subharmonic and unbounded above in $D_m$ but uniformly bounded above on that part of $\partial D_m$ which is in the finite plane. Put

$$M_m(r) = \sup_{z \in D_m} u_m(z), \quad |z| = r, \quad z \in D_m.$$

Then there is a positive constant $c$ such that for all sufficiently large $r$,

$$\sum_{i=1}^{p} (\log M_m(r))^{-1} \leq 2 (\log r - c)^{-1}. \quad (1)$$

(We note that if $u$ is subharmonic in a domain $D$ and $\zeta \in \partial D$, we define $u(\zeta) = \limsup_{z \to \zeta} u(z)$, $z \in D$.) A convenient reference is Lemma 3 in Domar [4].

We make the following assumptions on the domain $D$ and the analytic function $f$:

I. $D$ is an unbounded domain in the complex plane with at least one finite boundary point.

II. $f$ is analytic in $D$ and for every point $\zeta \in \partial D$, $\zeta \neq \infty$, we have

$$\limsup_{z \to \zeta} |f(z)| \leq 1, \quad z \in D. \quad (2)$$

**Theorem 1.** Let $D$ and $f$ be as above. We assume that $f(D) \cap \{ |w| > \sqrt{2} \} \neq \emptyset$. Then there is a constant $c$ such that for all sufficiently large $r$, we have

$$\log M(r, f) \geq \log r - c, \quad (3)$$

where $M(r, f) = \sup_{z \in D} |f(z)|$, $|z| = r, \ z \in D$. 

Corollary 1 (the "universal" Phragmén–Lindelöf theorem). Let \( f \) be analytic in \( D \) and assume that (2) holds at every finite boundary point \( \zeta \). If
\[
\lim \inf \frac{M(r, f)}{r} = 0, \quad r \to \infty,
\]
then \( |f(z)| \leq 1 \) throughout \( D \).

Proof of Corollary 1. The conclusion is trivial if \( f \) is bounded. If \( f \) is unbounded, there exists \( w_0 \in f(D) \) with \( |w_0| > \sqrt{2} \). But then inequality (3) and assumption (4) contradict each other. Consequently, \( f \) must be bounded and Corollary 1 is proved.

For other proofs of the Newman conjecture, see Fuchs [5], Gehring, Hayman and Hinkkanen [6] and Toppila [8].

Remark 1. Condition (4) in Corollary 1 is best possible. To see this, it suffices to consider the example \( D = \{ z : |z| > 1 \} \) and \( f(z) = z \).

Remark 2. If there are no finite boundary points in \( D \), \( f \) will be entire and our conclusions are trivial.

Corollary 2. Let \( D \) and \( f \) satisfy (I) and (II). Assume that there exists \( w_0 \in f(D) \) such that \( f^{-1}(w_0) \) contains at least \( p \) points. Then we have either
\[
\log M(r, f) \leq p \log r + O(1), \quad r \to \infty,
\]
or
\[
\log M(r, f) \geq (p + 1) \log r + O(1), \quad r \to \infty.
\]

Proof of Theorem 1. We first find \( z_0 \) such that \( f(z_0) = w_0, \, f'(z_0) \neq 0 \) and \( |w_0| > \sqrt{2} \). If \( h(z) = e^{\beta}(f(z) - w_0) \), we choose \( \beta \in \mathbb{R} \) in such a way that
\[
h'(z) \neq 0 \quad \text{if Re } h(z) = 0 \quad \text{or} \quad \text{Im } h(z) = 0,
\]
\[
\text{if } |f(z)| \leq (|w_0| + \sqrt{2})/(2\sqrt{2}), \quad \text{we have Re } h(z) > 0 \text{ and Im } h(z) > 0.
\]

It is clear that (8) is true for all \( \beta \) in an interval \( J \) and that (7) is true for almost all \( \beta \in J \).

Since \( |w_0| > \sqrt{2} \), \( f \) must be unbounded in \( D \). Continuing \( h^{-1} \) along the positive and negative real axis with \( h^{-1}(0) = z_0 \), we obtain two curves \( L_1 = L_1(\beta) \) and \( L_2 = L_2(\beta) \) in the domain of \( f \) which start at \( z_0 \). We claim that for almost all \( \beta \in J \), we have
\[
L_1 \cap L_2 = \{ z_0 \},
\]
\[
h^{-1}(w) \to \infty \quad \text{as } w \to \infty \text{ along the real axis}.
\]