AN APPLICATION OF THE AHLFORS DISTORTION THEOREM*

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1. The following uniqueness theorem is due to P. Malliavin (2, 5.5.1, p. 203).

Theorem A. Hypothesis. The function \( g(z) \) is holomorphic in the half-plane \( x \geq 0 \) and

\[
|g(z)| \leq (v(x)e^{-k(r)})^{x} \quad (z = x + iy = re^{i\theta}; \ x \geq 0)
\]

where

\( v(x) \) is continuous and \( v(x) \geq 1 \) in \( x \geq 0 \);

\[
k(r) > 0, \ 0 < k(R) - k(r) < A\{\log(R/r) + 1\} \quad (0 < r < R);
\]

and if

\[
S(t) = \sup_{x \geq 0} \{t - \log v(x)\}x
\]

then, for every \( a \geq 0 \)

\[
\int_{0}^{\infty} r^{-2}S[k(r) - a]dr = \infty.
\]

Conclusion.

\( g(z) \equiv 0. \)

Malliavin's statement of the Theorem is slightly different. The function

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$v(x)^*\text{ is replaced by a step-function and our } k(r) \text{ would be } k'(r) \text{ in his notation. Malliavin's form is easily derived from Theorem A.}$

Malliavin's proof is fairly intricate and uses ideas from functional analysis. In view of the many applications of Theorem A to problems of closure and of quasi-analyticity it may be worth while to give another proof of Theorem A.

The proof given in this paper is based on the following special case of the

**Ahlfors Distortion Theorem** (1, a proof can also be found in 3, p. 93–98). Let $D$ be the domain

$$|t| < \Delta(\sigma), \quad -\infty < \sigma < \infty$$

in the $s = \sigma + it$ plane. Let

$$w = u + iv = w(s)$$

map $D$ conformally on

$$|v| < \frac{1}{2}\pi, \quad -\infty < u < \infty$$

in such a way that $u \to +\infty$ as $s \to +\infty$ in $D$ and $u \to -\infty$ as $s \to -\infty$ in $D$.

If

$$s = \sigma + it \in D, \quad s_1 = \sigma_1 + it_1 \in D$$

and

$$\int_\sigma^{\sigma_1} \frac{d\lambda}{\Delta(\lambda)} > 4,$$

then

$$u(s_1) - u(s) > \frac{1}{2\pi} \int_\sigma^{\sigma_1} \frac{d\lambda}{\Delta(\lambda)} - 4\pi.$$

By means of the substitutions