MORE ON MONADIC LOGIC. PART C: MONADICALLY INTERPRETING IN STABLE UNSUPERSTABLE \( \mathcal{F} \) AND THE MONADIC THEORY OF \( \omega \lambda \)

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ABSTRACT
We continue investigating the strength of monadic logic in elementary classes. Mainly we show that all stable unsuperstable theories with finite vocabulary are either among the (easily definable class of) hopelessly complicated or are essentially as complicated as a variant of the tree \( \omega \mathbb{Z} \).

The main point here is the second section (see [BSh156]): if \( \mathcal{F} \) is (a first order) stable, not superstable theory with finite vocabulary ( = set of predicates and function symbols), then we can, in monadic logic, interpret in it essentially trees \( (\omega \mathbb{Z}, <) \) with quantification (Qpdf) (on pressing down functions). (Note: if \( (\mathcal{F}, 2\text{nd}) \leq (\mathcal{F}, \text{Mon}) \), this follows immediately as the class of such trees (up to isomorphism) is definable in second order logic, so the statement follows from 2.6.)

So this is another step in the classification of pairs \( (\mathcal{F}, Q) \), \( \mathcal{F} \) a first order theory, \( Q \) a quantifier. This, of course, raises the question of how complicated is the theory of such trees in \( L(Q^{pd}) \); this was dealt with in [Sh205], §1, where erroneously we said that the above interpretation appeared in [BSh156]. We give here a revised form of part of [Sh205], §1.

As for [Sh205], §2, note that conjecture 2.14A (on ultrafilters on \( \omega \)) was

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disproved in Blass–Shelah [BISh242], but its aim — doing the interpretation in ZFC — was accomplished in Part B; i.e. [Sh284b] (but not interpreting the second order theory of $\lambda$ — only after the forcing).

The referee points out another wrong quotation of a nonexisting theorem which was corrected by strengthening 2.1 (omitting a saturation demand).

**NOTATION.** The letter $T$ serves as a tree and as a first order theory for the latter meaning, so we use here $\mathcal{T}$.

§1. Remarks on [Sh205], §1

**CONVENTION.** Hence, 1. $x$ refers to [Sh205], § 1 (or the revised/additional ones here).

On the connection between the $L(Q^{\omega})$-theory of $K^\omega_{tr}$ (trees with $\omega$ levels) and of the trees $\omega \rightarrow \lambda$ ($pd$ stands for pressing down) note:

1.3(5). **NOTATION.** $K^\omega_{tr,\lambda} = \{ T \in K^\omega_{tr} : (\forall x \in T)[|Suc_T(x)| = \lambda] \}$, $K^\omega_{tr,\hom} = \bigcup_{\lambda} K^\omega_{tr,\lambda}$.

1.3B. **NOTATION.** (a) $T_1$ is a nice subtree of $T_0$ (both in $K^\omega_{tr}$) if
   (i) $(T_1, \leq \tau_1)$ is a submodel of $(T_0, \leq \tau_0)$.
   (ii) for every $x \in T_1$,
   \[ Suc_{T_1}(x) = \{ y \in T_1 : x < y, \land (\exists z \in T_1)[x < z < y] \} \]
   is a front of $T_0$, above $x$, i.e.
   \[ (\forall z \in T_0)[x < z \rightarrow (\exists y \in Suc_{T_1}(x))[z \leq y \lor y \leq z]]. \]

   Every branch (= maximal linearly ordered subset) of $T_0$ to which $x$ belongs, is not disjoint to $Suc_{T_1}(x)$.

1.4. **CLAIM.** (1) The $L(Q^{\omega})$-theories of $K^\omega_{tr}$ and $\{ \omega \rightarrow \lambda : \lambda \geq \kappa_0 \}$ (i.e. of $K^\omega_{tr,\hom}$) are recursive one in the other.
   (2) Being a nice subtree is a transitive relation.
   (3) Let $T \in K^\omega_{tr}$.
      (a) for every tree $T$ for some nice subtree $T_1$ of $T$ and $\lambda$, $T_1 \equiv \omega \rightarrow \lambda$.
      (4) Every nice subtree of $\omega \rightarrow \lambda$ is isomorphic to $\omega \rightarrow \lambda$, so every nice subtree of a tree from $K^\omega_{tr,\hom}[K^\omega_{tr,\lambda}]$ belongs to $K^\omega_{tr,\hom}[K^\omega_{tr,\lambda}]$.

   **PROOF.** Straightforward: first prove (2), (3), then deduce (1) (by the more elaborate Claim 1.4A below).

   In fact we can say how much of the model theory is preserved.