CANTOR–BENDIXSON DEGREES AND CONVEXITY IN $\mathbb{R}^2$

BY

MENACHEM KOJMAN

Department of Mathematics and Computer Science
Ben Gurion University of the Negev, Beer Sheva 84105, Israel
e-mail: kojman@cs.bgu.ac.il

ABSTRACT

We present an ordinal rank, $\delta^3$, which refines the standard classification of non-convexity among closed planar sets. The class of closed planar sets falls into a hierarchy of order type $\omega_1 + 1$ when ordered by $\delta$-rank.

The rank $\delta^3(S)$ of a set $S$ is defined by means of topological complexity of 3-cliques in the set. A 3-clique in a set $S$ is a subset of $S$ all of whose unordered 3-tuples fail to have their convex hull in $S$. Similarly, $\delta^n(S)$ is defined for all $n > 1$.

The classification cannot be done using $\delta^2$, which considers only 2-cliques (known in the literature also as “visually independent subsets”), and in dimension 3 or higher the analogous classification is not valid.

1. Introduction

Let $S$ be a set in a linear space and suppose that $S$ is not convex. One would like to measure how far $S$ is from being convex. The most natural number for measuring non-convexity of a set $S$ is the least number of convex subsets of $S$ needed to cover $S$. Let us, then, define $\gamma(S)$ as the least cardinality of a collection of convex sets whose union equals $S$. The function $\gamma$ is adopted as the basic measurement of non-convexity. Classification by $\gamma$ gives countably many different classes of sets with finite $\gamma$ and (potentially) only two classes with infinite $\gamma$: sets with countable $\gamma$ and sets with uncountable $\gamma$.

In this paper we define for each $n > 1$ a degree functions $\delta^n$, and show that $\delta^3$ refines the $\gamma$-classification for closed, planar sets. The class

Received April 8, 1998
The first step in understanding the structure of a set $S$ with $\gamma(S) = \lambda$ is to understand why $S$ fails to decompose into a union of fewer than $\lambda$ convex sets. There is an easy sufficient condition for $S$ not to be a union of fewer than $\lambda$ convex sets: the existence of a subset $P \subseteq S$ of cardinality $\lambda$, with the property that for any two points in $P$ the line segment connecting them is not contained in $S$. No two of those points can sit in the same convex subset of $S$, hence $S$ is not a union of $n$ convex sets. Call a subset of $S$ with this property “visually independent”. Let $\alpha(S)$ be the supremum of cardinalities of all visually independent subsets in $S$.

Does $\alpha$ measure non-convexity adequately? This can be rephrased as whether there exists a “reasonable” function $f$ so that $\gamma(S) \leq f(\alpha(S))$.

For general sets this is badly false (see [6], Section 5), and also in “nice” sets in dimension 3 or higher the connection between $\alpha$ and $\gamma$ is not well behaved. Nevertheless, closed sets in $\mathbb{R}^2$ show some tight connections between $\alpha$ and $\gamma$. A long sequence of results [4, 9, 1, 2] culminated in the discovery [3] that $\gamma(S) \leq f(\alpha(S))$ for some function $f$, for closed planar sets. Later it was shown that $f$ is at most $n^6$ in [8]. Recently, $n^6$ was lowered to $18n^3$ by Matousek and Valtr in [7], where also a lower bound of $O(n^2)$ was set.

In sets which are not a finite union of convex sets, the connection between $\alpha$ and $\gamma$ is not as tight. There exist compact, planar sets with countable $\alpha$ and uncountable $\gamma$ ([6], Example 2.1). Put differently, the class of closed planar sets with countable $\alpha$ contains also sets with uncountable $\gamma$. This means that the notion of visual independence does not capture all the information as to why a closed $S \subseteq \mathbb{R}^2$ cannot be covered by countably many convex subsets.

However, a generalization of visual independence does. Call a subset $P \subseteq S$ a 3-clique if every 3-element subset $X \subseteq P$ satisfies that its convex closure is not contained in $S$. Theorem 2.2 in [6] says that a closed set in the plane is not a countable union of convex sets if, and only if, it contains an uncountable (actually perfect) 3-clique. Namely, the only reason for such $S$ not to be a countable union of convex sets is that it contains an uncountable 3-clique.

Since, by this theorem, the full information about non-convexity of a closed planar set is stored in the collection of its 3-cliques, it is natural to try and classify non-convexity of such sets by classifying their 3-cliques. It turns out that the standard topological classification of closed countable sets — the Cantor–Bendixson degree — indeed works: for every closed planar set which is a countable union of