SCALING ESTIMATES FOR SOLUTIONS AND DYNAMICAL LOWER BOUNDS ON WAVEPACKET SPREADING

By

DAVID DAMANIK* AND SERGUEI TCHEREMCHANTSEV

Abstract. We establish quantum dynamical lower bounds for discrete one-dimensional Schrödinger operators in situations where, in addition to power-law upper bounds on solutions corresponding to energies in the spectrum, one also has lower bounds following a scaling law. As a consequence, we obtain improved dynamical results for the Fibonacci Hamiltonian and related models.

1 Introduction

Consider a discrete one-dimensional Schrödinger operator,

$$[H\psi](n) = \psi(n + 1) + \psi(n - 1) + V(n)\psi(n),$$

on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{1, 2, \ldots\}$. In the case of $\ell^2(\mathbb{Z}_+)$, we work with a Dirichlet boundary condition, $\psi(0) = 0$, but our results easily extend to all other self-adjoint boundary conditions.

A number of recent papers (e.g., [DST, DT, JL1, JL2, JSS, KKL, T1, T2]) were devoted to proving lower bounds on the spreading of an initially localized wavepacket, say $\psi = \delta_1$, under the dynamics governed by $H$, typically in situations where the spectral measure of $\delta_1$ with respect to $H$ is purely singular and sometimes even pure point.

A standard quantity that is considered to measure the spreading of the wavefunction is the following: For $p > 0$, define

$$\langle |X|_p^p \rangle(T) = \sum_n |n|^p a(n, T),$$

where

$$a(n, T) = \frac{2}{T} \int_0^{\infty} e^{-2t/T} |\langle e^{-itH} \delta_1, \delta_n \rangle|^2 dt.$$
Clearly, the faster $\langle |X|_{\delta_1}^p \rangle (T)$ grows, the faster $e^{-itH} \delta_1$ spreads out, at least averaged in time. Typically, one wants to prove power-law lower bounds on $\langle |X|_{\delta_1}^p \rangle (T)$; hence it is natural to define, for $p > 0$, the lower growth exponent $\beta_{\delta_1}^-(p)$ by

$$\beta_{\delta_1}^-(p) = \liminf_{T \to +\infty} \frac{\log \langle |X|_{\delta_1}^p \rangle (T)}{\log T}.$$ 

When one wants to bound these exponents from below for specific models, it is useful to connect these quantities to the qualitative behavior of the solutions of the difference equation

$$(4) \quad u(n + 1) + u(n - 1) + V(n)u(n) = Eu(n)$$

for energies $E$ in the spectrum of the operator $H$. In fact, most of the known results are based on such a correspondence; compare [JL1, JL2] for an approach where the link is furnished by Hausdorff-dimensional properties of spectral measures, and [DST, DT] for a direct approach without intermediate step. The latter papers use power-law upper bounds on solutions corresponding to energies from a set $S$ to derive lower bounds for $\beta_{\delta_1}^-(p)$. The set $S$ can even be very small. One already gets non-trivial bounds when $S$ is not empty. If $S$ is not negligible with respect to the spectral measure of $\delta_1$, the bounds are stronger; but there are situations of interest (e.g., random polymer models [JSS]) where the spectral measure assigns zero weight to $S$.

While both approaches yield bounds on $\beta_{\delta_1}^-(p)$ for all $p > 0$, in concrete applications there is a transition point, $p_0$, such that the method from [JL1, JL2] works better for $0 < p < p_0$, whereas the method from [DST, DT] gives better bounds for $p > p_0$.

Our goal here is to develop an approach that, whenever it applies, gives stronger lower bounds than both previous methods throughout the entire range of the powers $p$.

A model for which the exponents $\beta_{\delta_1}^-(p)$ have been heavily studied (e.g., [D, DKL, DST, DT, JL2, KKL]) is given by the Fibonacci Hamiltonian. This is the standard model of a one-dimensional quasicrystal, and it is one of the few for which one can actually prove “anomalous” transport properties rigorously; compare the discussion in [KKL]. With this model in mind, we refine the results from [DST, DT, KKL] in what follows. It turns out that the bounds on $\beta_{\delta_1}^-(p)$ can be considerably strengthened if, in addition to power-law upper bounds for solutions, one also assumes suitable lower bounds. The necessary input does in fact hold for the Fibonacci model and related ones, as we show, and we thereby obtain improved dynamical results which are strictly better than all previously