THE DISTRIBUTION FUNCTION OF THE CONVOLUTION SQUARE OF A CONVEX SYMMETRIC BODY IN $\mathbb{R}^n$

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ABSTRACT

By analyzing the distribution function of the convolution square of a convex and symmetric body we obtain some affine invariants related to the body. These invariants have a geometric interpretation.

Introduction and notations

The starting point of our investigation is a paper of K. Kiener [K]. Before we explain his results we have to introduce some notation. Let $C$ be a convex body in $\mathbb{R}^n$ (i.e. $C$ is a compact convex subset of $\mathbb{R}^n$ with non-empty interior). By $I_C$ we denote the indicator function of $C$; the convolution square of $C$ is defined by $F = I_C * I_C$ (we will also investigate the function $G = I_C * I_{-C}$ which in the case of a symmetric body coincides with $F$). The distribution function of $F$ is given by

$$V_F(\delta) = \text{Vol}_n([F > \delta]) = \text{Vol}_n \{ x \in \mathbb{R}^n : F(x) > \delta \}$$

where $\text{Vol}_n$ denotes the $n$-dimensional Lebesgue measure. By a volume preserving linear transformation we mean a linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det T = 1$. In [K] Kiener proved the following theorem:

Let $C$ be a convex body in $\mathbb{R}^n$. Choose $\alpha > 0$ such that $\text{Vol}_n(C) = \text{Vol}_n(\alpha B_2^n)$ (where $B_2^n$ denotes the euclidean ball of radius 1). If the distribution function of the convolution square coincides with that of $\alpha B_2^n$ then $C$ is an ellipsoid, i.e. $C$ is an image of $\alpha B_2^n$ under a volume preserving linear transformation.

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A crucial point in proving this theorem was the following formula

$$\lim_{\delta \to 0} \frac{V_F(\delta)}{\text{Vol}_n(C)} (\text{Vol}_n(C) - \delta) = \text{Vol}_n(P^*)$$

where $V_F$ denotes as above the distribution function of the convolution square of $C$ and $P^*$ denotes the polar of the projection body of $C$. We deduce this formula from an exponential bound for the convolution square. We also analyze the behavior of $V_F(\delta)$ for symmetric convex bodies as $\delta$ tends to zero. It turns out that there is an analogy between certain bodies associated with the convolution square of a convex symmetric body and the so-called floating bodies. The corresponding results for the floating bodies were obtained by V.D. Milman and M. Gromov [G.M], C. Schütt and E. Werner [S.W] and C. Schütt [S].

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The convolution square

Let $C$ be a convex symmetric body in $\mathbb{R}^2$ and let $\text{pr}_1 C = [-c,c]$ denote the projection of $C$ to the first coordinate. Define

$$f : [-c,c] \to \mathbb{R} \text{ by } f(x) = \sup \{y : (x,y) \in C\}.$$ 

Then $f$ is concave and

$$C = \{(x,y) \in \mathbb{R}^2 : x \in [-c,c], -f(-x) \leq y \leq f(x)\}.$$

For $\lambda \geq 0$ set

$$x_\lambda := \max \{x \geq 0 : (x,f(x)) \in \partial C \cap (\partial C + \lambda(0,1))\}.$$

**LEMMA 1:** Let $\lambda, \lambda_0 \geq 0, \lambda + \lambda_0 \leq 2f(0)$. Then

$$0 \leq x_{\lambda_0} - x_{\lambda_0 + \lambda} \leq \lambda \frac{x_{\lambda_0}}{2f(0) - \lambda_0}.$$

**Proof:** For $t \geq 0, x_t$ satisfies the equation

$$f(x_t) = -f(-x_t) + t.$$ 

Since the function $F(x) := f(x) + f(-x)$ is concave and symmetric, the right hand side follows immediately. The left hand side of the inequality follows from the fact that $F$ is decreasing on $[0,c].$