Leserforum

Coherent Risk Measures *

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1. Introduction and notation

The concept of coherent risk measures with its axiomatic characterization was introduced in the paper [ADEH1] and further developed in [ADEH2] and [D2]. This paper emphasizes the use of coherent risk measures as premium principles. In this respect it is related to the work of [Wan]. Throughout the paper, we will work with a probability space \((\Omega, \mathcal{F}, P)\). With \(L^{\infty}(\Omega, \mathcal{F}, P)\) (or \(L^{\infty}(P)\) or \(L^{\infty}\)), we mean the space of all equivalence classes of bounded real valued random variables. The space \(L^{0}(\Omega, \mathcal{F}, P)\) (or \(L^{0}(P)\) or \(L^{0}\)) denotes the space of all equivalence classes of real valued variables. The space \(L^{0}\) is equipped with the topology of convergence in probability. The space \((L^{\infty}(P), \mathcal{F}, P)\) or \(L^{0}(P)\) or \(L^{0}\)) is the dual space of the space of integrable (equivalence classes of) random variables, \(L^{1}(\Omega, \mathcal{F}, P)\) (also denoted by \(L(P)\) or \(L^{1}\)). We will identify, through the Radon-Nikodym theorem, finite measures that are absolutely continuous with respect to \(P\), with their densities, i.e. with functions in \(L^{1}\).

2. Definition and characterization

In this section we recall, without proofs, the main theorems of the papers [ADEH1], [ADEH2] and [D2]. In our setting the definition of a coherent risk measure as given in [ADEH1] can be written as:

**Definition 2.1.** A mapping \(\varphi: L^{\infty}(\Omega, \mathcal{F}, P) \to \mathbb{R}\) is called a coherent risk measure if the following properties hold

1. If \(X \geq 0\) then \(\varphi(X) \leq 0\).
2. Subadditivity: \(\varphi(X_{1} + X_{2}) \leq \varphi(X_{1}) + \varphi(X_{2})\).
3. Positive homogeneity: for \(\lambda \geq 0\) we have \(\varphi(\lambda X) = \lambda \varphi(X)\).
4. For every constant function \(a\) we have that \(\varphi(a + X) = \varphi(X) - a\).
5. The Fatou property: \(\varphi(X) \leq \lim \inf \varphi(X_{n})\), for any sequence, \((X_{n})_{n \geq 1}\), of functions, uniformly bounded by 1 and converging to \(X\) in probability.

**Remark.** We refer to [ADEH1] and [ADEH2] for an interpretation and discussion of the above properties. Property (4) relates money of today with money of tomorrow. The reader can convince him/herself that by the usual process of discounting, well known in actuarial sciences, the case of nonzero interest can be reduced to the present case. See also [ADEH2] for the proper details and a discussion about these. Property (5) is a continuity property that allows us to avoid so called finitely additive measures. For details about these developments see [D2]. To see a coherent risk measure as an insurance premium, let us analyse the following situation. An insurance company wants to underwrite a policy which has a possible payout described by the nonnegative random variable \(X\). Suppose for the moment that this random variable is bounded. The extension to arbitrary random variables will be given later. If we only look at this situation, the future worth of this contract is then given by \(-X\).

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In order to be able to underwrite this policy, the company has to add money to this. This is of course the premium to be paid by the insured. In our language this means that the premium is given by \( \varphi(-X) \), which according to our requirements is positive. The use of coherent risk measures as premiums is not well spread, but as we will see later, it is quite natural.

**Remark.** If \( \varphi \) is a coherent risk measure and if we put \( \psi(X) = \varphi(-X) \) we get a translation invariant submodular functional. If we put \( \varphi(X) = -\varphi(X) \), we obtain a submodular functional. These objects are well known and were used in different fields of mathematics. E.g. submodular functionals were studied by Choquet in connection with the theory of capacities, see [Ch]. We refer the reader to [Wa1] for the development and the application of the theory to imprecise probabilities and belief functions. These concepts are certainly not disjoint from risk management considerations. In [Wa2], P. Walley gives a discussion of properties that may also be interesting for risk measures. In [Maa], Maaß gives an overview of existing theories. The following properties of a coherent risk measures are immediate

1. \( \varphi(0) = 0 \) since by positive homogeneity: \( \varphi(2 \cdot 0) = 2 \varphi(0) \).
2. If \( X \leq 0 \), then \( \varphi(X) \geq 0 \). Indeed \( 0 = \varphi(X + (-X)) \leq \varphi(X) + \varphi(-X) \) and if \( X \leq 0 \), this implies that \( \varphi(X) = \varphi(-X) \geq 0 \).
3. If \( X \leq Y \) then \( \varphi(X) \geq \varphi(Y) \).
4. \( \varphi(a) = -a \) for constants \( a \in \mathbb{R} \).
5. If \( a \leq X > b \), then \( -a \geq \varphi(X) \geq -b \).
6. \( \varphi(X - \varphi(X)) = 0 \).

**Remark.** The interpretation of these properties is immediate.

The following result is basic in the theory of coherent risk measures. For a proof we refer to [D2].

**Theorem.** There is a one-to-one correspondence between

1. coherent risk measures \( \varphi \) having the Fatou property,
2. closed convex sets of probability measures \( \mathcal{P}_\alpha \subset L^1(P) \).

The correspondence is given by

\[
\varphi(X) = \sup_{Q \in \mathcal{P}_\alpha} \mathbb{E}_Q[-X].
\]

**Remark.** The interpretation as a premium calculation principle is clear. We first go back to the old practice in life insurance where, more or less, premiums are calculated as expected values. It is only more or less, since for different policies different mortality tables, i.e. different probability measures, are used. In the setting of coherent risk measures, we take expected values with respect to several probability measures. Then we take the highest of all these outcomes. This measure was denoted as WCM\(_\alpha\) in [ADEH2]. Of course, if the space is atomless, it doesn’t matter if we use the condition \( P[A] \geq \alpha \) instead of the strict inequality \( P[A] > \alpha \). It can be shown that for random variables with continuous distribution function, the risk measure is given by

\[
\varphi(X) = \mathbb{E}_P[-X \mid X \leq q_\alpha(X)],
\]

where \( q_\alpha(X) \) denotes the \( \alpha \)-quantile of \( X \), i.e. satisfies \( P[X \leq q_\alpha(X)] = \alpha \).

**Example 3.** This example shows how to introduce higher moments in coherent risk measures. For fixed \( p > 1 \) and \( \beta > 1 \), we consider the weakly compact convex set:

\[
\mathcal{P}_\alpha = \left\{ Q \left| \frac{dQ}{d\mathbb{P}} \right|_p \leq \beta \right\}.
\]

If \( p = \infty \) and \( \beta = 1/\alpha \), then we simply find back the preceding example. So we will suppose that \( 1 < p < \infty \).

If we define \( q = \frac{p}{p - 1} \), the conjugate exponent, then we have the following result:

**Theorem:** For nonnegative bounded functions \( X \), we have that

\[
c \|X\|_q \leq \varphi(-X) \leq \beta \|X\|_q,
\]

where \( c = \min(1, \beta - 1) \).