Laurent–Padé approximants to four kinds of Chebyshev polynomial expansions.  
Part II: Clenshaw–Lord type approximants

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Laurent–Padé (Chebyshev) rational approximants \( P_m(w, w^{-1})/Q_n(w, w^{-1}) \) of Clenshaw–Lord type \([2,1]\) are defined, such that the Laurent series of \( P_m/Q_n \) matches that of a given function \( f(w, w^{-1}) \) up to terms of order \( w^{\pm(m+n)} \), based only on knowledge of the Laurent series coefficients of \( f \) up to terms in \( w^{\pm(m+n)} \). This contrasts with the Maehly-type approximants \([4,5]\) defined and computed in part I of this paper \([6]\), where the Laurent series of \( P_m \) matches that of \( Q_n f \) up to terms of order \( w^{\pm(m+n)} \), but based on knowledge of the series coefficients of \( f \) up to terms in \( w^{\pm(m+2n)} \). The Clenshaw–Lord method is here extended to be applicable to Chebyshev polynomials of the 1st, 2nd, 3rd and 4th kinds and corresponding rational approximants and Laurent series, and efficient systems of linear equations for the determination of the Padé–Chebyshev coefficients are obtained in each case. Using the Laurent approach of Gragg and Johnson \([4]\), approximations are obtainable for all \( m \geq 0 \), \( n \geq 0 \). Numerical results are obtained for all four kinds of Chebyshev polynomials and Padé–Chebyshev approximants. Remarkably similar results of formidable accuracy are obtained by both Maehly-type and Clenshaw–Lord type methods, thus validating the use of either.

Keywords: Chebyshev series, Laurent series, Laurent–Padé approximant, Chebyshev–Padé approximant, Clenshaw–Lord approximant, end-point singularities

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1. Introduction
Chebyshev polynomials of 1st, 2nd, 3rd and 4th kinds, namely \( T_n(x) \), \( U_n(x) \), \( V_n(x) \) and \( W_n(x) \) of degree \( n \) are defined \([6]\) by

\[
\begin{align*}
T_n(x) &= \cos n\theta, \\
U_n(x) &= \frac{\sin(n+1)\theta}{\sin \theta}, \\
V_n(x) &= \frac{\cos(n+(1/2))\theta}{\cos(1/2)\theta}, \\
W_n(x) &= \frac{\sin(n+(1/2))\theta}{\sin(1/2)\theta},
\end{align*}
\]

(1)

where \( x = \cos \theta \) for \(-1 \leq x \leq 1\) and \( 0 \leq \theta \leq \pi \).
A detailed discussion of the theory, properties, application and relative advantages of these four polynomials is given in [7], and includes rational approximation and series expansion.

2. Clenshaw–Lord type first kind approximants

Recall [1,2,7] that a (first kind) Clenshaw–Lord Padé–Chebyshev approximant $E'_pT'(x)/E'qT'(x)$ to a given infinite series $E'cT(x)$ is defined by

$$\frac{\sum_{r=0}^{m} p_r T_r(x)}{\sum_{s=0}^{n} q_s T_s(x)} = \sum_{i=0}^{m+n} c_i T_i(x) + O(T_{m+n+1}(x)),$$

where the dash denotes that the first term of the sum is halved. Writing

$$T_r(x) = \frac{1}{2}(w^r + w^{-r}), \quad \text{where} \quad x = \frac{1}{2}(w + w^{-1}),$$

then

$$\frac{\sum_{r=0}^{m} p_r \frac{1}{2}(w^r + w^{-r})}{\sum_{s=0}^{n} q_s \frac{1}{2}(w^s + w^{-s})} = \frac{1}{2} \sum_{i=0}^{m+n} c_i (w^r + w^{-r}) + O(w^{m+n+1}, w^{-m-n-1}). \quad (3)$$

By using the Laurent form, Gragg and Johnson [4] extended the Clenshaw–Lord method to the case $m < n$, and we can adopt the same approach.

Now defining the denominator, as in [1], by

$$\frac{1}{2} \sum_{s=0}^{n} q_s (w^s + w^{-s}) = \frac{1}{2} \lambda \sum_{r=0}^{n} b_r w^r \sum_{t=0}^{n} b_t w^{-t} \quad (4)$$

and setting $q_0 = 1$ and $b_0 = 1$ we may substitute (4) into (3) to give:

$$\frac{\sum_{r=0}^{m} \frac{1}{2} p_r (w^r + w^{-r})}{\frac{1}{2} \lambda \sum_{s=0}^{n} b_s w^s} = \sum_{r=0}^{n} b_r w^r \sum_{t=0}^{m+n} c_t \frac{1}{2} (w^r + w^{-r}) + \cdots + O(w^{m+n+1}, w^{-m-n-1}). \quad (5)$$

Equating coefficients of $w^k$ for $k > m$:

$$0 = \sum_{r=0}^{m} b_r (c_{k-r} + c_{r-k}) \quad (k = m + 1, \ldots, m + n),$$

where only one term in $c_0$ is included for $k = r$ (and $r = k$), i.e.

$$0 = \sum_{r=0}^{m} b_r c_{|k-r|} \quad (k = m + 1, \ldots, m + n), \quad (6)$$