On Linear Gravitational Theories.

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Summary. - It is shown that linear metrical gravitational theory based on one tensor field of second order which satisfies the weak-equivalence principle is equivalent to, in linear approximation, general theory of relativity.

It is well known that the vacuum field equations of general relativity in the case of linear approximation can be represented as

\[ Dh_{ij} = \lambda \eta_{ij} \]

in suitable co-ordinate conditions (Hilbert conditions), where

\[ h_{ij} = g_{ij} - \eta_{ij} \]

is the small perturbation of the flat metric \( \eta_{ij} \), \( \lambda \) is the cosmological constant and \( D \) is the D'Alembertian operator.

In the theory of relativity such important effects like the motion of ultrarelativistic particles (gravitational waves, gravitational light deflection, Shapiro effects, etc.) are connected with the analysis of the solution of eq. (1). So, it is of interest to investigate the question of conditions under which eq. (1) is unique.

To investigate the above question we shall prove the following theorem:

Theorem. Any linear Lorentz invariant theory of gravitation based on one tensor field of second order which satisfies the weak equivalence principle is equivalent to the linear approximation of general relativity.

Proof. If we use a tensor field of the second order then, in general, the field equation in linear approximation may be written in the form

\[ A\eta_{ij} + z_1 h_{ij} + z_2 h_{ij} + z_3 h_{ij} + z_4 (h_{j,ik} + h_{i,jk}) + z_5 h^{ki,l}_{ij} + z_6 h^{kl}_{ijkl} \eta_{ij} = T_{ij} \]

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where $A$ is a constant, $T_{ij}$ is the energy-momentum tensor, and the raising and lowering of indices are done by $\eta_{ij}$. In view of the weak-equivalence principle one has in linear approximation

\[(3) \quad T_{i}^{\,j} = T_{j}^{\,i} = 0.\]

In view of eq. (3), eq. (2) yields

\[(4) \quad \alpha_1 h_{,i} + \alpha_2 h_{,j} + \alpha_3 D h_{,i} + \alpha_4 D h_{,k} + \alpha_5 D h_{,i,k} + \alpha_6 D h_{,i,k} + \alpha_7 h_{,k} = 0.\]

It can easily be seen that eq. (4) will hold if

\[(5) \quad \alpha_1 + \alpha_3 D = 0, \quad \alpha_2 + \alpha_4 D = 0, \quad \alpha_4 + \alpha_6 D + \alpha_7 = 0.\]

In the linear theory it is possible to choose $\alpha$'s as a polynomial function of $D$:

\[(6) \quad \alpha_i = a_i + b_i D + c_i D^2 + \ldots\]

Thus eq. (5) yields

\[(7) \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = -b_1, \quad a_4 = -b_2, \quad a_7 = -a_8, \ldots\]

Now we use eq. (1) and eq. (7) to transform eq. (2). A straightforward calculation leads one to obtain in vacuum

\[(8) \quad (A - 4a_3 \lambda + a_6 \lambda) \eta_{ij} = a_3 h_{,ij} - a_6 (h_{,ik} + h_{,ij}) - a_5 h_{,k} = 0.\]

And, for further simplification of this equation, we use four arbitrary co-ordinate conditions which we formulate in the form

\[(9) \quad h_{ik} = \beta h_{,i}\]

By virtue of the co-ordinate conditions (9), eq. (8) takes the form

\[(10) \quad (A - 2a_3 \lambda + a_6 \lambda) \eta_{ij} = 0.\]

It is evident that eq. (10) is satisfied if and only if

\[(11) \quad \lambda = A/(2a_3 - a_6), \quad 2a_3 \neq a_6.\]

Hence it is possible to choose such a $\lambda$ so that eqs. (1) and (2) are equivalent in the co-ordinate system satisfying conditions (9). This proves the theorem.

Now it is necessary to consider the two peculiar cases:

a) Case I: $a_6 = 0$,

b) Case II: $2a_3 = a_6$. 